STOCHASTIC PROCESS OF PRECIPITATION

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P. Todorovic and V. Yevjevich

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LIST OF SYMBOLS

Symbol	Meaning
ε	is an element of
	is contained in, contains
$\{a,b,c,\ldots\}$	the set consisting of the elements a,b,c...
$A \cup B$	the union of sets A and B
$A \cap B$	the intersection of sets A and B
$\bigcup_\nu A_\nu$	the union of sets A_1, A_2, A_3,...
$\bigcap_\nu A_\nu$	the intersections of sets A_1, A_2, A_3,...
A^c	the complement of A
θ	the empty set (impossible event)
Ω	the space of elementary event
$\omega, (\omega \varepsilon \Omega)$	an elementary event
$\mathcal{A}$	σ-field
$\mathcal{B}$	Borel field
P	probability measure
ξ_t	the rainfall intensity at some instant of time t
X_t	total amount of precipitation in some interval of time (t_o,t)
η_t	number of storm periods in (t_o,t)
t_o	moment of time when observations begin
$\tau_\nu - t_o$	the elapsed time up to the end of ν-th storm period
X_ν	the total amount of precipitation during exactly ν storm periods
Z_ν	total amount of precipitation during ν-th storm period
$E_\nu^{t_o,t}$	the event that exactly ν complete storm periods will occur in (t_o,t)
$G_\nu^{x_o,x}$	the events that total amount of precipitation during ν storms will be less or equal $(x-x_o)$
$\int_{t_o}^{t} \lambda_1(s)ds$	the average number of storms in (t_o,t)
$F_\nu(t)$	the distribution function of τ_ν
$f_\nu(t)$	the density function of τ_ν
$A_\nu(x)$	the distribution function of X_ν
$a_\nu(x)$	the density function of X_ν
$B_\nu(z)$	the distribution function of Z_ν
$b_\nu(z)$	the density function of Z_ν
$\Gamma(\nu)$	Gamma function
$F_t(x)$	is the distribution function of X_t for all $t \geq t_o$
$f_t(x)$	is the density function of X_t
$E(X)$	is the mathematical expectation of the random variable X
$\{\xi_t; t \geq t_o\}$	represents a stochastic process or a family of random variables

ABSTRACT

The stochastic process of precipitation intensity, $\xi_t \geq 0$, as a time series at a given precipitation station, is presented. Six random variables are used as descriptors of the ξ_t-process: (1) the number of storms in a time interval; (2) the maximum number of storms after a reference time, with their total precipitation not exceeding a given value; (3) the lapse time between a reference time and the end of a storm; (4) the total precipitation for ν storms; (5) the total precipitation of the ν-th storm; and (6) the total precipitation in a given interval of time.

Two parameters are shown to be important in deriving the probability distributions of the above six descriptors: λ_1, the average number of storms per unit time interval (in the text designated as the density of storms in time); and λ_2, the yield characteristic of storms (in the text defined as the inverse of the average water yield per storm). λ_1 and λ_2 are periodic functions of time, with the year as the period, as illustrated with four examples (Durango, Colorado; Fort Collins, Colorado; Austin, Texas and Ames, Iowa precipitation time series, the first three using daily precipitation values and the last one hourly values).

Two definitions of storms have been imposed by the data available: (1) every rainy day or every rainy hour is considered as a storm; and (2) the uninterrupted sequence of rainy days or rainy hours is considered as a storm. In studying λ_1- and λ_2-parameters, it was shown for rainy days that the first definition gives a larger number and the second definition a smaller number of storms per time interval than the expected true number of storms.

The number of storms in an interval is Poisson-distributed, with the parameter in that distribution being the integral of λ_1-parameter over that interval.

The comparison of the theoretical probability density function for the total precipitation of ν storms (ν = 1, 2, 3 and 15 in the first definition and ν = 1, 2, 3 and 10 in the second definition) and their empirical frequency density curves shows a good agreement for the four examples investigated, considering the inevitable sampling errors.

The comparison of the theoretical probability distribution functions of the lapse time for ν-th storm (ν = 1, 2, 3 and 15 in the first definition, and ν = 1, 2, 3 and 10 in the second definition of storms) and their empirical frequency distribution curves show a good agreement for the four examples investigated.

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P. Todorovic* and V. Yevjevich**

Chapter I

INTRODUCTION

1.1 General character of hydrologic time series. The analysis of hydrologic time series and other hydrologic sequences by the appropriate mathematical models, that describe either the patterns in sequence of a river flow or precipitation or their spacial distributions, represent an important step in predicting characteristics of future water supply and planning of water resource projects. Among the various concepts that have been used in the analysis of hydrologic processes one can distinguish two basically different approaches, deterministic and probabilistic. In the following, two examples are outlined that point out the distinction between these two conceptual approaches.

A hydrologic (or generally speaking, a physical) phenomenon is subject to some laws that govern its evolution. A physical phenomenon is assumed to be a deterministic one if, on the basis of the present state, the future characteristics of the phenomenon can be predicted with certainty (are sure outcomes). For instance, the Newton laws of motions are deterministic in the sense that on the basis of the given present state of a moving body the future states are uniquely determined. Similarly, as an example in hydrology, the outflow hydrograph from an impervious surface is a deterministic process, if the rainfall input is assumed to be known as its distribution over the surface and in a given time interval, and the evaporation is negligible.

The laws of random phenomena, for example, those that govern the evolution of rainfall phenomenon in time, are stochastic in the sense that on the basis of the present state only probabilities of the future outcomes may be determined. For the precipitation random process, if η_t stands for the number of bursts or storms in the interval of time (o,t), which gives the present state of the phenomenon, the number of bursts or storms $\eta_{t,t+\Delta t}$ in the interval of time $(t,t+\Delta t)$, can never be predicted with certainty for any $\Delta t > 0$. In other words, $\eta_{t,t+\Delta t}$ is a random variable defined over some probability space $(\Omega,\mathcal{A},P)$ for every $\Delta t > 0$. Since $\eta_{t,t+\Delta t}$ is a discrete random variable, only probabilities

$$P_\nu(t,\Delta t) = P\{\eta_{t,t+\Delta t} = \nu\}, \quad \nu = 0,\ 1,\ 2,\ \dots\ ,$$

where $\sum_{\nu=o}^{\infty} P_\nu(t,\Delta t) = 1$ for all $t > o$ and $\Delta t > o$, may be determined.

This study of precipitation phenomenon follows in principle the probabilistic approach. With respect to the nature of the phenomenon, this approach is the most logical for the analysis and prediction of the future characteristics of the time series of precipitation. To summarize, the precipitation phenomenon is considered from the aspect of the theory of stochastic processes. A stochastic process is a mathematical abstraction of an empirical process, which in this case is a physical phenomenon evolving in time and governed by probabilistic laws. A stochastic process is a random variable X_t, that depends on time t, or a family of random variables, one for each instant of time, t, defined on a probability space.

1.2 Structure of hydrologic time series. Past experience with various studies of time series of hydrologic phenomena, such as evaporation, runoff, precipitation, and others has pointed to their three basic characteristics expressed in the form of time series components:

(a) The secular or long term variations conceived as fluctuations of the basic characteristics of time series (distribution function, mathematical expectation, variance, extreme values, etc.) in function of time, either as the regular persistence of cyclicity and trends, or as unspecified changes of non-stationary character. These variations are often referred to as "climatic changes" or "secular components of geophysical time processes."

(b) The periodic component related to the astronomical cycle of the day, or the periodic component related to astronomical cycle of the year, are usually defined as "periodic movements."

(c) The stochastic components that are the results of the probabilistic nature of the phenomena considered are frequently called "stochastic variations or fluctuations."

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Among the most controversial questions in the current hydrologic investigations is the problem of the existence of secular components, that is the existence of periodicity, trends or other non-stationarity in the probability structure of hydrologic time series of annual values, beyond the periodicity of the year. Non-stationarity in secular components may be only in some characteristic parameters of the series, such as the trend in the mathematical expectation, in the variance, covariance, or in the higher moments. In other words, there is the question whether or not the "secular components" or "secular non-stationarity" do really exist in time series.

Some studies do not support the concept of non-stationarity in hydrologic series of annual values. At least the most reliable information available in the last 100-150 years of thousands of annual precipitation and annual runoff series [1] does not show significant trends and periodicities. However, this subject is still not closed and will likely be treated often in the future as more data become available and as better methods for the discrimination and testing of various properties of time series are developed. This problem of eventual long-term non-stationarity of time series is not treated in this study. It is assumed the present-day techniques of time series analysis do not, and eventually cannot, discern any significant non-stationarity in the series of annual values of basic hydrologic phenomena. In other words, for a couple of centuries (1-2 preceding and 1-2 next centuries), it is assumed for the purposes of this study that no secular component of non-stationary character is present in the series of annual values of natural hydrologic processes.

The time series of variables which refer to any interval, smaller than a year (seasonal, monthly, daily, hourly, and so forth) and the continuous time series of intensity of a hydrologic phenomenon exhibit both periodic components and stochastic components. Therefore, they are basically non-stationary time series.

With respect to the within-the-year periodicity, it seems logical to conclude, by physical considerations and as well as by experience, that some hydrologic time series must have the periodical-probability structure. At least they have periodicity in some parameters. The probabilistic nature of hydrologic time series is a result of mutual interactions of an immense number of various physical causative factors. In spite of very regular and deterministic astronomical movements, many hydrologic time series are extremely irregular and their behavior is unpredictable because of the dominance of probabilistic part in the phenomenon.

1.3 Subject of this investigation. The intermittent stochastic process of precipitation intensity ξ_t, which is the rainfall intensity at the instant t, is the subject matter of this investigation. The total precipitation in a small interval of time $(t, t+\Delta t)$ is approximately equal to $\Delta t \cdot \xi_s$, where $t < s < t + \Delta t$. Therefore, the total precipitation X_t in a time interval (o,t), if this limit exists, is given by

$$X_t = \lim_{n\to\infty} \sum_{i=1}^{n} \xi_{s_i} \cdot \Delta t_i = \int_o^t \xi_s \, dt , \tag{1.1}$$

where for all $i = 1, 2, \ldots, n$, and $\Delta t_i = \frac{t}{n}$ and $(i-1)\Delta t < s_i < i\Delta t$. If X_t is a differential function of time t, then

$$\xi_t = \frac{dX_t}{dt} \tag{1.2}$$

where ξ_t represents a non-negative stochastic process or a non-negative family of random variables

$$\{\xi_t ;\ t \in T\} \tag{1.3}$$

where T is the domain of definition of the process. On the basis of eq. (1.1), X_t represents a stochastic process as well. Both ξ_t and X_t are continuous parameter random processes and since $\xi_t \geq 0$ for all $t \varepsilon T$, it follows that sample functions of X_t are non-decreasing functions of t.

The process ξ_t is highly intermittent because for any time interval $(t_1,t_2) \subset T$, the probabilities that $\xi_t = 0$ and $\xi_t > 0$ are positive, i.e.,

$$P\{\xi_t = 0,\ t \varepsilon (t_1,t_2)\} > 0 \text{ and } P\{\xi_t > 0,\ t \varepsilon (t_1,t_2)\} > 0.$$

If $(t_2 - t_1) \to \infty$, then

$$P\{\xi_t = 0,\ t \varepsilon (t_1,t_2)\} \to 0 \text{ and } P\{\xi_t > 0,\ t \varepsilon (t_1,t_2)\} \to 0.$$

Phenomenologically speaking, $\{\xi_t = 0,\ t \varepsilon (t_1,t_2)\}$ represents the event that $\xi_t = 0$ in (t_1,t_2). In other words there will be no precipitation during the time interval (t_1,t_2). The condition $P\{\xi_t = 0,\ t \varepsilon (t_1,t_2)\} \to 0$ if $(t_2 - t_1) \to \infty$ means that the probability of the event, that after time t_1 it will never rain again, is zero. Similarly, $\{\xi_t > 0,\ t \varepsilon (t_1,t_2)\}$ denotes the event that during (t_1,t_2) it continuously rains. The condition $P\{\xi_t > 0,\ t \varepsilon (t_1,t_2)\} \to 0$ if $(t_2 - t_1) \to \infty$ means that the probability is zero that after the time t_1 it will rain continuously.

The (instantaneous) precipitation intensity ξ_t is rarely measured or published as such. Instead, usually the ξ_t process is given as integrated total precipitation or as average intensities over unit time intervals (10 minutes, 30 minutes, one hour, two hours, and so on). Therefore, in practice instead of a continuous record of ξ_t only values in a finite set of time units are available.

Those properties of intermittent precipitation that are connected to the stochastic process ξ_t are subject of this study. In other words, various characteristics of rainfall that can be obtained from the recorded precipitation data are expressed as functions of the process ξ_t. The areal distribution of precipitation or any physical phenomenon in the atmosphere, which affects the precipitation, are not parts of this investigation. In the sequel, the process ξ_t is referred to as the basic stochastic process of precipitation.

1.4 Research objectives. The main objective of the study is to present a mathematical model for investigating those properties of precipitation that are related to the stochastic process ξ_t. Such properties are the number of storms in a given interval of time, the total precipitation during the given number of storm periods, and similar. It is a result of the general characteristics of probability theory that these properties of rainfall may be derived from ξ_t-process. In other words, and mathematically speaking, a series of functions of random variables, defined here as functions of ξ_t, are considered and their probability structures are determined as stochastic processes. Only one-dimensional distribution functions of these functions of random variables are subject of this study.

It should be stressed here that the study of the process $X_t = \int_0^t \xi_s \, ds$ is undertaken and not of the process ξ_t itself. The reason is it is much simpler to study the monotonically increasing process X_t than the basic process ξ_t.

1.5 Two fundamental approaches to investigation of hydrologic stochastic processes. The basic hydrologic processes in time domain contain daily and within-the year periodicities. Two fundamental approaches may be used in these cases:

(a) A process is transformed in such a way as to remove the periodic components, and then to investigate the remaining stationary stochastic process. This approach is feasible for continuous processes or those derived processes of discrete nature which are not intermittent, say $\xi_t > 0$.

(b) A process is considered as observed, with its periodic part unseparated from the stochastic part, but many functions of the process (derived variables) exhibit the periodicity. This second approach is attractive for intermittent processes, like storm precipitation, or storm flows of intermittent rivers. In this second approach, the density of storms per unit time interval, or the intensity of storms (storm yield) per unit time interval, will show a periodicity, if the density of storms and the water yield per storm are functions of seasons. This second approach is the one taken in this study.

CHAPTER II

THEORETICAL BACKGROUND

2.1 Definitions and notations. As was mentioned before, the precipitation intensity $\xi_t \geq 0$ is the basic process of the study. It is apparent that $\xi_t = 0$, whenever there is no precipitation, and $\xi_t > 0$ when it rains or snows. If t_o stands for the time when observations of the rainfall phenomenon begin, and ξ_{t_o} is the rainfall intensity at that instant, then with respect to the nature of precipitation, it is not possible to predict with certainty the value of ξ_t at any instant of time after t_o; i.e., ξ_t is a random variable for any $t > t_o$. For example, if an arbitrary sequence of times $t_1, t_2, \ldots, t_\nu$ are selected from (t_o, ∞), then it is not possible to anticipate precipitation intensities at these instants. In other words,

$$\xi_{t_1}, \xi_{t_2}, \ldots, \xi_{t_\nu} \tag{2.1}$$

are random variables. Since it is valid for any sequence of instants from (t_o, ∞), it is apparent that ξ_t is an infinite family of random variables, or one variable for each $t \varepsilon (t_o, \infty)$, which is denoted in the following manner

$$\{\xi_t;\ t > t_o\} \ . \tag{2.2}$$

This family of random variables represents the basic continuous parameter stochastic processes of this study.

Generally speaking, random variables of (2.1) have different distribution functions, i.e.,

$$F_{t_1}(x) = P\{\xi_{t_1} \leq x\},\ F_{t_2}(x) = P\{\xi_{t_2} \leq x\}, \ldots$$

and

$$F_{t_1}(x) \neq F_{t_2}(x) \neq , \ \ldots \tag{2.3}$$

The corresponding mathematical expectations

$$E(\xi_{t_1}) = \int x dF_{t_1}(x);\quad E(\xi_{t_2}) = \int x dF_{t_2}(x), \ \ldots$$

are different for different t_i (see Fig. 1), as well. One can choose another sequence of instants $t'_1, t'_2, \ldots t'_n$ and determine mathematical expectations of $\xi_{t'_1}$, $\xi_{t'_2} \ldots \xi_{t'_n}$, etc. In this manner a curve is obtained, (Fig. 1), which represents mathematical expectations, $E(\xi_t)$ of the stochastic process ξ_t. In a similar way one can obtain the variance σ_t^2 of ξ_t as a function of time t defined as

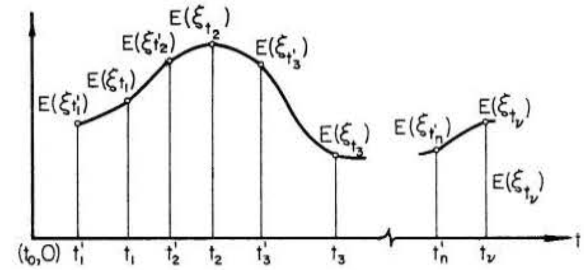

Fig. 1 Graphical presentation of mathematical expectation of random variables, ξ_{t_k}

$$\sigma_t^2 = E[\xi_t - E(\xi_t)]^2 \tag{2.4}$$

If one assumes ξ_t to have the periodic-probability structure, with both $E(\xi_t)$ and σ_t being periodic functions of time t and with regard to the oscillation inside the day or inside the year, then $E(\xi_t)$ and σ_t follow the periodic movement of the day or the year, respectively.

In the moment theory of stochastic processes, it is sometimes useful, instead of the process ξ_t, to consider its linear transformation ε_t, defined as

$$\varepsilon_t = \frac{\xi_t - E(\xi_t)}{\sigma_t} \ . \tag{2.5}$$

In that case the process ξ_t is expressed as

$$\xi_t = \sigma_t \varepsilon_t + E(\xi_t) \ ,$$

i.e., the stochastic process ξ_t is separated in two components, the stochastic part, ε_t, and the deterministic parts, $E(\xi_t)$ and σ_t. It is easy to see that

$$E(\varepsilon_t) = 0 \quad \text{and} \quad E(\varepsilon_t^2) = 1 \ , \quad \text{for all } t \geq t_o.$$

Since no assumptions have been made about the stochastic process ξ_t, nothing specific can be said about ε_t. It may be stationary in the wide or narrow sense, or it may be a stochastic process of any kind.

The intermittency of the precipitation intensity process enables the application of a particular conceptual structure, namely precipitation bursts or storms as intermittent successive events in the process. A storm is defined as continuous precipitation between two non-rainy intervals, even though the total amount of precipitation and the duration of some storms may be very small. Therefore, there is a difference between the colloquial definition and concept of a storm, and the definition of the storm in the probabilistic sense of discrete precipitation storm events. Each has a different duration, a different total precipitation, and a different shape of storm intensities. This latter definition will be used in this text, while the everyday concept of storms will be left for those storm events which have relatively significant total precipitation.

A schematic representation of a sample function of the process $\{\xi_t;\ t \geq t_o\}$ is given in Fig. 2. It is easy to single out many characteristics of this process as separate stochastic processes. For instance, the beginning times, the centroids, the mid-points, or the ending time of individual storm events are particular properties of the process. Similarly, the total precipitation of each event, the total precipitation from t_o to t, the duration of each event, the maximum intensity of each event, etc., are stochastic processes which depend on the basic process ξ_t. Choosing which of the various stochastic processes, derived from ξ_t, to study will depend on the problem at hand. Therefore, some arbitrary selection of random variables is made here for the study of derived stochastic processes. For some other processes, not studied here, it is logical to apply derivations similar to those shown in the following text.

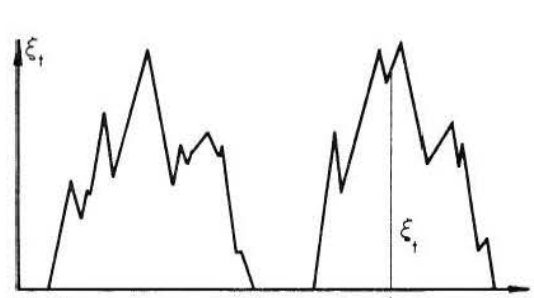

Fig. 2 Sample function of the process ξ_t, which is a precipitation hyetograph.

The first stochastic process derived from ξ_t is a linear functional (or linear random function) of ξ_t, defined as

$$X_t = x_o + \int_{t_o}^{t} \xi_s \, ds \qquad (2.6)$$

where s is a dummy integration variable, and represents the total precipitation for time t; x_o is total precipitation for time t_o and $\int_{t_o}^{t} \xi_s \, ds$ for time $(t-t_o)$. Since $\xi_t \geq 0$, it follows that for all $\Delta t > 0$

$$X_{t+\Delta t} - X_t = \int_{t}^{t+\Delta t} \xi_s \, ds \geq 0 \qquad (2.7)$$

and thus, sample functions of stochastic process $\{X_t;\ t > t_o\}$ are nondecreasing functions of time t (Fig. 3).

In further exposition, the following system of stochastic processes, defined as functions of ξ_t, will be discussed or studied (see Fig. 3):

(1) η_t, the number of complete storm events in the time interval $(t_o, t]$;

(2) η_x, the maximum number of storm events after t_o, with the total precipitation which does not exceed the quantity $x - x_o$;

(3) $\{\tau_\nu;\ \nu = 1, 2, \ldots,\}$, the times of ends of storm events, which is a random sequence of points on the time scale;

(4) X_ν, the total precipitation for ν storm events;

(5) $Z_\nu = X_\nu - X_{\nu-1}$, the total precipitation during ν-th storm event; and

(6) $\{X_t;\ t > t_o\}$, the total precipitation in the interval $(t_o, t]$ from the initial absolute time t_o to any time t, as a stepwise nondecreasing cumulative function of ξ_t.

Before going further, the discussion of some basic properties of stochastic processes from (1) to (6) is appropriate. Since t and x are continuous variables, η_t and η_x are continuous parameter stochastic processes. However, for fixed t and x, random variables, η_t and η_x, are of the discrete type (counting variables) i.e., η_t and η_x can be 0, 1, 2, ... only.

The stochastic processes τ_ν, X_ν and Z_ν are the discrete parameter stochastic processes; however, for the fixed value of the parameter ν, the processes τ_ν, X_ν and Z_ν are continuous random variables. By virtue of the definition, for all $\Delta t > 0$ and $\Delta x > 0$, $\eta_t \leq \eta_{t+\Delta t}$, and $\eta_x \leq \eta_{x+\Delta x}$; and for all $\nu = 1, 2, \ldots,$ and $k = 1, 2, \ldots.,$ $\tau_\nu \leq \tau_{\nu+k}$ and $X_\nu \leq X_{\nu+k}$. Finally, for all $\nu > 0$, $Z_\nu > 0$, and $X_\nu = \sum_{k=1}^{\nu} Z_k$.

2.2 Some basic considerations. In terms of probability theory, a random experiment or a random observation is the system

$$(\Omega = \{\omega\}, \mathcal{A}, P) \qquad (2.8)$$

where Ω is the space of elementary events ω, $\mathcal{A}$ is a σ-field or σ-algebra which consists of subsets of Ω, and P is probability measure defined on the class $\mathcal{A}$.

Phenomenologically speaking, an elementary event ω represents an outcome of the experiment and Ω represents the set of all possible outcomes of this experiment. The class $\mathcal{A}$ is the set of all possible events that are sets of ω and whose probability can be

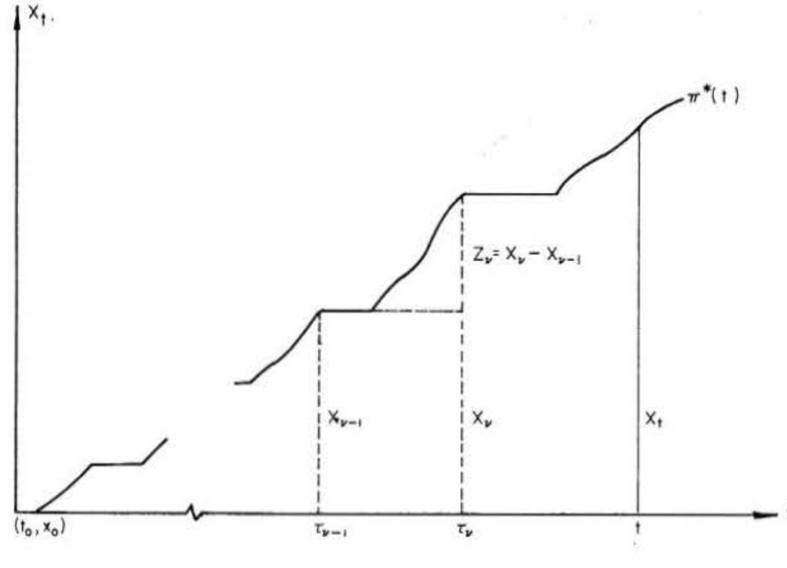

Fig. 3 A sample function of the process $\{X_t;\ t > t_o\}$

determined. Finally, P is a function (so called probability measure) which associates a definite probability to each event from $\mathcal{A}$.

In the case of precipitation, the intensity hyetograph of the type given in Fig. 2 represents an outcome ω of random observations, and Ω is the set of all possible hyetographs that may occur. Therefore, Ω represents a functional space, whose elements are functions of time t. One may assume Ω as the set of all continuous functions defined on (t_o,T), where T can be ∞. Any function from Ω is called a sample function or a realization of the stochastic process, ξ_t.

All possible measurable subsets of Ω(i.e., abusing a bit of language, all events whose probability calculation has sense) make the σ-field. System $(\Omega,\mathcal{A},P)$ is frequently called the probability space. Examples of measurable subsets or events belonging to the σ-field, assuming that Ω is the set of all sample functions of stochastic process ξ_t of precipitation intensity, are: (a) A subset of sample functions of Ω which at a time t have the cumulative values X_t smaller than a given x, where $x > x_o$; (b) the complementary subset of all functions that have $X_t \geq x$ at a given t; (c) the subset which satisfies the condition that the total number of storm events, η_t, in the interval (t_o,t) is $\eta_t < \nu$; (d) its complementary subset of $\eta_t \geq \nu$, and similar subsets.

The points τ_j, j = 1, 2, ... , which define the ends of storm events, may be replaced by the points of storm event beginnings, or storm events centroids, by their mid-points (50% of the total storm precipitation is before and 50% is after the mid-points). The basic results will not be changed if one set or another set of τ-points is selected for the definition of storm event positions in time. The end effect in sample functions is important in this case. The times t_o and t may divide storm events into two parts. For the interval $(t_o,t]$, the end point τ_ν is then inside the interval $(t_o,t]$, and the end point $\tau_{\nu+1}$ is outside this interval. It can be assumed these two end events may, on the average, compensate each other. Their parts inside the interval $(t_o,t]$, immediately after t_o and immediately before t, should constitute one storm event in such a way that ν end points τ_j, $j = 1, 2, \ldots,\nu$, in the interval $(t_o,t]$ are equivalent to ν storm events.

In order to determine some probabilistic properties of the above six families of random variables derived from ξ_t, of interest in this investigation, two classes of measurable subsets of Ω are first defined. In fact, the σ-algebra $\mathcal{A}$ is generated by the sets of these families.

First, let $E_\nu^{t_o,t}$ denote the subset of the space Ω which consists of all sample functions (all outcomes ω, or all realizations) of the stochastic process ξ_t, which have exactly ν points τ_j in $(t_o,t]$. In other words, $E_\nu^{t_o,t}$ represents the random event that exactly ν complete storms will occur in $(t_o,t]$. According to the definition

$$E_\nu^{t_o,t} = \{\tau_\nu \leq t < \tau_{\nu+1}\},\ \nu = 0,1,2,\ldots;\ t \geq t_o\ . \quad (2.9)$$

It is not difficult to see that for a fixed $t \geq t_o$ the system of random events of (2.9) represents a countable partition of the space Ω , i.e.,

$$\bigcup_{\nu=0}^{\infty} E_\nu^{t_o,t} = \Omega,\ E_i^{t_o,t} \cap E_j^{t_o,t} = \theta,\ i \neq j = 0,1,2,\ldots\ , \quad (2.10)$$

where θ stands for an empty set (the impossible event). In other words, the union of all events of (2.9) is a certain event, since in any time interval $(t_o,t]$ the number of storms must be 0, or 1, or 2,..., and any of these two events are disjoint events. However, it can be seen that in the general case

$$E_i^{t_o,t} \cap E_j^{t_o,t+\Delta t} \neq \theta \text{ for any } \Delta t > 0.$$

Second, let $G_\nu^{x_o,x}$ be the set (subset of Ω) of all sample functions of the stochastic process ξ_t, so that the total amount of precipitation X_ν during ν storms is less or equal to $(x - x_o)$, and $X_{\nu+1}$ exceeds $(x - x_o)$, i.e.,

$$G_\nu^{x_o,x} = \{X_\nu \leq x < X_{\nu+1}\}, \; \nu = 0,1,2,\ldots; \; x \geq x_o . \quad (2.11)$$

According to the definition, for all $x \geq x_o$, the system of random events of (2.11) has the following properties

$$\bigcup_{\nu=0}^{\infty} G_\nu^{x_o,x} = \Omega, \; G_i^{x_o,x} \cap G_j^{x_o,x} = \theta, \; i \neq j = 0,1,2,\ldots;$$

$$x \geq x_o \; , \quad (2.12)$$

i.e., the system of (2.11) represents a countable partition of the space Ω.

The reason the systems of (2.9) and (2.11) of random events are selected is because the distribution and density functions of the above six stochastic processes, derived from ξ_t, can be expressed in simple forms as functions of probabilities of $E_\nu^{t_o,t}$ and $G_\nu^{x_o,x}$. On the other hand, probabilities $P(E_\nu^{t_o,t})$ and $P(G_\nu^{x_o,x})$ can be determined easily for all $\nu = 0,1,2,\ldots, t \geq t_o$ and $x \geq x_o$, under very general assumptions.

Now, it is possible to define precisely the structure of the probability space $(\Omega,\mathcal{A},P)$. The definition of the space Ω is clear, as it is the set of all sample functions of ξ_t. The σ-field (or σ-algebra) $\mathcal{A}$ is generated by the class of sets $\{X_\nu \leq x\}$ and $\{\tau_\nu \leq t\}$ where $\nu = 1,2,\ldots, x \geq x_o, t \geq t_o$. It is apparent for $\nu = 0,1,2,\ldots, E_\nu^{t_o,t} \varepsilon \mathcal{A}$. Indeed, according to the definition

$$E_\nu^{t_o,t} = \{\tau_\nu \leq t < \tau_{\nu+1}\} = \{\tau_\nu \leq t\} \cap \{\tau_{\nu+1} > t\} \, \varepsilon \, \mathcal{A} .$$

Similarly,

$$G_\nu^{x_o,x} = \{X_\nu \leq x < X_{\nu+1}\} = \{X_\nu \leq x\} \cap \{X_{\nu+1} > x\} \varepsilon \, \mathcal{A} .$$

Therefore, the classes of sets, $E_\nu^{t_o,t}$ and $G_\nu^{x_o,x}$, generate the σ-field $\mathcal{A}$.

Finally, two basic properties of probability measure P are assumed to be

$$\lim_{\Delta t \to o} \frac{\sum_{\nu=2}^{\infty} P(E_\nu^{t,t+\Delta t})}{\Delta t} = 0 \text{ or } \lim_{\Delta t \to o} \frac{1-P(E_o^{t,t+\Delta t})-P(E_1^{t,t+\Delta t})}{\Delta t}$$

$$= 0 \quad (2.13)$$

and

$$\lim_{\Delta x \to o} \frac{\sum_{\nu=2}^{\infty} P(G_\nu^{x,x+\Delta x})}{\Delta x} = 0 \text{ or } \lim_{\Delta x \to o} \frac{1-P(G_o^{x,x+\Delta x})-P(G_1^{x,x+\Delta x})}{\Delta x}$$

$$= 0 \quad (2.14)$$

for any value t and x, respectively. In other words, the sum of all probabilities, $P(E_\nu^{t,t+\Delta t})$ for $\nu \geq 2$, is a higher order infinitesimal quantity in comparison with Δt, and similarly for the sum of $P(G_\nu^{x,x+\Delta x})$ for $\nu \geq 2$ in comparison with Δx, if Δt and Δx tend to zero.

Assumption of (2.13) means physically the probabilities of two or more storms to occur in the infinitesimally small interval of time Δt tend to zero much faster than Δt. Similar physical interpretation is valid for the assumption of (2.14).

In addition, the conditional probabilities for the occurrence of $E_1^{t,t+\Delta t}$, given $E_\nu^{t_o,t}$, should satisfy the following condition

$$\boxed{\lambda_1(t,\nu) = \lim_{\Delta t \to o} \frac{P(E_1^{t,t+\Delta t} | E_\nu^{t_o,t})}{\Delta t}} \quad (2.15)$$

with $\lambda_1(t,\nu) \geq 0$ for all points $t\varepsilon(t_o,\infty)$. To determine the probability of a storm event just after t, in the interval $(t,t+\Delta t)$, the function $\lambda_1(t,\nu)$ should be integrated in this finite small interval, or approximately it is $\lambda_1(t,\nu)\Delta t$.

Similarly, the conditional probability for the occurrence of $G_1^{x,x+\Delta x}$, given $G_\nu^{x_o,x}$, should satisfy the following condition

$$\boxed{\lambda_2(x,\nu) = \lim_{\Delta x \to o} \frac{P(G_1^{x,x+\Delta x} | G_\nu^{x_o,x})}{\Delta x}} \quad (2.16)$$

with $\lambda_2(x,\nu) \geq 0$ for all points of $x\varepsilon(x_o,\infty)$. The conditional probability of a storm of Δx-amount at x is for small Δx approximately equal to $\lambda_2(x,\nu)\Delta x$.

In the following, it will be assumed the functions $\lambda_1(t,\nu)$ and $\lambda_2(x,\nu)$ do not depend on ν, i.e., when $\lambda_1(t,\nu) \equiv \lambda_1(t)$ and $\lambda_2(x,\nu) \equiv \lambda_2(x)$. It is easy to see $\lambda_1(t)$ and $\lambda_2(x)$ are constants if ξ_t is a stationary stochastic process. It is expected that $\lambda_1(t)$ and $\lambda_2(x)$ are periodic functions for the studied ξ_t process, if this process contains the periodic components.

2.3 Distribution and density functions of derived stochastic processes. The one-dimensional distribution (and corresponding density) functions of the stochastic processes derived previously from ξ_t will be determined in this subchapter.

Distributions of η_t, η_x, τ_ν, X_ν and Z_ν can be easily expressed in terms of probabilities of $E_\nu^{t_o,t}$ - and $G_\nu^{x_o,x}$ - events as follows:

By definition

$$P\{\eta_t = \nu\} = P(E_\nu^{t_o,t}) \ . \tag{2.17}$$

Denoting the mathematical expectation

$E(\eta_t)$ by $\Lambda_1(t_o,t)$, then

$$\Lambda_1(t_o,t) = \sum_{\nu=1}^{\infty} \nu P(E_\nu^{t_o,t}) \tag{2.18}$$

Similarly, for the process η_x,

$$P\{\eta_x = \nu\} = P(G_\nu^{x_o,x}) \ , \tag{2.19}$$

and the mathematical expectation $E(\eta_x)$ is

$$\Lambda_2(x_o,x) = \sum_{\nu=1}^{\infty} \nu P(G_\nu^{x_o,x}). \tag{2.20}$$

For $P\{\tau_\nu \leq t\} = F_\nu(t)$ and $dF_\nu(t)/dt = f_\nu(t)$, the distribution and the density function of the variable τ_ν, it can be proven (see Appendix A) for every $\nu = 1, 2, \ldots$

$$F_\nu(t) = 1 - \sum_{j=o}^{\nu-1} P(E_j^{t_o,t}), \tag{2.21}$$

and

$$f_\nu(t) = \lambda_1(t,\nu-1)P(E_{\nu-1}^{t_o,t}), \tag{2.22}$$

provided the conditions of (2.13) and (2.15) are satisfied.

For $P\{X_\nu \leq x\} = A_\nu(x)$ and $a_\nu(x)$, the respective distribution and density functions of the variable X_ν, it can be proven (see Appendix A) that for every $\nu = 1, 2, \ldots$

$$A_\nu(x) = 1 - \sum_{j=o}^{\nu-1} P(G_j^{x_o,x}) \tag{2.23}$$

and

$$a_\nu(x) = \lambda_2(x,\nu-1)P(G_{\nu-1}^{x_o,x}) \ , \tag{2.24}$$

provided the conditions of (2.14) and (2.16) are satisfied.

For $P\{Z_\nu \leq z\} = B_\nu(z)$ and $b_\nu(z)$ the distribution and density functions of the random variable Z_ν, it can be proven, keeping in mind $Z_\nu = X_\nu - X_{\nu-1}$, under the assumption $Z_1, Z_2, \ldots, Z_\nu, \ldots$ are independent random variables,

$$a_\nu(x_o+z) = \int_o^z a_{\nu-1}[(x_o+z)(1-\frac{u}{z})]\ b_\nu(u)\ du, \tag{2.25}$$

where u is the dummy integration variable. Solution of this convolution integral equation gives the density function $b_\nu(z)$ of Z_ν .

It is clear from eqs. (2.17) through (2.25) that distributions and density functions of η_t, η_x, τ_ν, X_ν and Z_ν may be obtained as soon as probabilities of the events $E_\nu^{t_o,t}$ and $G_\nu^{x_o,x}$ are known.

2.4 Determination of probabilities of $E_\nu^{t_o,t}$ - and $G_\nu^{x_o,x}$ -events. The probability that ν storm events will occur in the interval $(t_o, t+\Delta t)$, where $\Delta t > 0$, is

$$P(E_\nu^{t_o,t+\Delta t}) = \sum_{r=o}^{\nu} P(E_{\nu-r}^{t_o,t} \cap E_r^{t,t+\Delta t})$$

By using the condition of (2.13), then

$$P(E_\nu^{t_o,t+\Delta t}) = P(E_\nu^{t_o,t} \cap E_o^{t,t+\Delta t}) + P(E_{\nu-1}^{t_o,t} \cap E_1^{t,t+\Delta t})$$

$$+ \sigma(\Delta t) \ ,$$

where $\sigma(\Delta t)$ is the higher order infinitesimal in comparison with Δt, when Δt approaches zero.

As

$$E_o^{t,t+\Delta t} = \Omega - \bigcup_{\tau=1}^{\infty} E_\tau^{t,t+\Delta t} \ ,$$

then

$$P(E_\nu^{t_o,t+\Delta t}) - P(E_\nu^{t_o,t}) = - P(E_\nu^{t_o,t} \cap E_1^{t,t+\Delta t}) +$$

$$+ P(E_{\nu-1}^{t_o,t} \cap E_1^{t,t+\Delta t}) + \sigma(\Delta t) \ .$$

Taking into account the condition of (2.15), one obtains the differential equations for the probability of events $E_\nu^{t_o,t}$

$$\frac{P(E_\nu^{t_o,t})}{\partial t} = -\lambda_1(t,\nu)P(E_\nu^{t_o,t}) + \lambda_1(t,\nu-1)P(E_{\nu-1}^{t_o,t}) \tag{2.26}$$

for $\nu = 1, 2, \ldots,$ and

$$\frac{\partial P(E_o^{t_o,t})}{\partial t} = -\lambda_1(t,0)P(E_o^{t_o,t}) \tag{2.27}$$

for $\nu = 0$.

Finally, by virtue of the relation of (2.15), it follows

$$\lambda_1(t) = \sum_{\nu=o}^{\infty} \lambda_1(t,\nu)P(E_\nu^{t_o,t}) \qquad (2.28)$$

where

$$\lambda_1(t) = \lim_{\Delta t\to 0} \frac{P(E_1^{t,t+\Delta t})}{\Delta t} . \qquad (2.29)$$

If $\lambda_1(t,\nu)$ is an integrable function for all $\nu = 0, 1, 2, \ldots$, and

$$\lim_{t\to t_o} P(E_\nu^{t_o,t}) = \begin{cases} 0 \text{ for } \nu > 0 \\ 1 \text{ for } \nu = 0 \end{cases}$$

then the solution of eq. (2.26) is equal to

$$P(E_\nu^{t_o,t}) = \exp\{-\int_{t_o}^{t} \lambda_1(s,0)ds\}\cdot \int_{t_o}^{t} \lambda_1(t_1,\nu-1)$$

$$\exp\{\int_{t_o}^{t_1} \Delta\lambda_1(s,\nu)ds\}\cdot\int_{t_o}^{t_1} \lambda_1(t_2,\nu-2)\exp\{\int_{t_o}^{t_2}\Delta\lambda_1(s,\nu-1)ds\}$$

$$\ldots \int_{t_o}^{t_{\nu-2}}\lambda_1(t_{\nu-1},1)\exp\{\int_{t_o}^{t_{\nu-1}}\Delta\lambda_1(s,2)ds\}\cdot\int_{t_o}^{t_{\nu-1}}\lambda_1(t_\nu,0)$$

$$\exp\{\int_{t_o}^{t_\nu} \Delta\lambda_1(s,1)ds\}\cdot dt_\nu dt_{\nu-1} \ldots dt_1 \qquad (2.30)$$

where $\Delta\lambda_1(s,\nu-1) = \lambda_1(s,\nu-i) - \lambda_1(s,\nu-i-1)$, and s, t_1, t_2, ..., t_ν are dummy integration variables.

Some particular cases of eq. (2.30) are considered here.

1. $\underline{\lambda_1(t,\nu)}$ <u>is independent of ν</u>, or $\lambda_1(t,\nu) \equiv \lambda_1(t)$. In that case obviously $\Delta\lambda_1(s,\nu-1) = 0$, or $\Delta\lambda_1(t) = 0$, so that

$$P(E_\nu^{t_o,t}) = e^{-\int_{t_o}^{t} \lambda_1(s)ds} \int_{t_o}^{t} \lambda_1(t_1) \int_{t_o}^{t_1}\lambda_1(t_2) \ldots$$

$$\int_{t_o}^{t_{\nu-1}}\lambda_1(t_\nu)\, dt_\nu dt_{\nu-1} \ldots dt_1 . \qquad (2.31)$$

It is easy to see eq. (2.31) may be reduced to the form

$$\boxed{P(E_\nu^{t_o,t}) = e^{-\int_{t_o}^{t} \lambda_1(s)ds} \frac{[\int_{t_o}^{t} \lambda_1(s)ds]^\nu}{\nu!} .} \qquad (2.32)$$

2. $\underline{\lambda_1(t,\nu) = \xi(\nu)}$, <u>or independent of t but dependent on ν</u>. Since the difference $\Delta\lambda_1(t,\nu-1)$ is now equal to $\Delta\lambda_1(t,\nu-1) = \zeta(\nu-1) - \zeta(\nu-i-1)$, assuming $t_o = 0$, eq. (2.30) becomes

$$P(E_\nu^{t_o,t}) = e^{-\zeta(\nu)t}\prod_{i=o}^{\nu-1} \zeta(i) \int_{o}^{t} e^{\Delta\zeta(\nu)t_1} \int_{o}^{t_1} e^{\Delta\zeta(\nu)t_2}\ldots$$

$$\int_{o}^{t_{\nu-1}} e^{\Delta\zeta(\lambda)t_\nu} dt_\nu dt_{\nu-1} \ldots dt_1 \qquad (2.33)$$

where $\Delta\zeta(\nu) = \zeta(\nu) - \zeta(\nu-1)$. If one assumes $\zeta(0)$, $\zeta(1)$, ... represent an arithmetic progression, then $\Delta\zeta(\nu) = \rho$ = constant, and

$$P(E_\nu^{t_o,t}) = \frac{\prod_{i=o}^{\nu-1} \zeta(i)}{\nu!\rho^\nu} (e^{\rho t} - 1)^\nu e^{-\xi(\nu)t} , \qquad (2.34)$$

which finally gives

$$\boxed{P(E_\nu^{t_o,t}) = e^{-\zeta(o)t} \frac{\Gamma[\frac{\zeta(o)}{\rho} + \nu]}{\Gamma(\nu+1)} (1 - e^{-\rho t})^\nu .} \qquad (2.35)$$

3. $\underline{\lambda_1(t,\nu) = \lambda_1(t)\cdot\zeta(\nu)}$. In this case the differences of eq. (2.30) become $\Delta\lambda_1(t,\nu-1) = \lambda_1(t)\cdot[\zeta(\nu-i) - \zeta(\nu-i-1)]$. Assuming $t_o = 0$, and $\zeta(o)$, $\zeta(1)$, ..., represent the arithmetic progression with $\Delta\zeta(\nu) = \rho$ = constant for all $\nu = 1, 2, \ldots$, then

$$\boxed{P(E_\nu^{t_o,t}) = e^{-\zeta(o)\int_o^t \lambda_1(s)ds} \cdot\frac{\Gamma[\frac{\zeta(o)}{\rho} + \nu]}{\Gamma(\nu+1)}\left(1-e^{-\rho\int_o^t \lambda_1(s)ds}\right)^\nu .} \qquad (2.36)$$

In a similar way as for $P(E_\nu^{t_o,t})$, the properties of $P(G_\nu^{x_o,x})$ can be derived. By using the condition (2.14) and (2.16), then

$$\frac{\partial P(G_\nu^{x_o,x})}{\partial x} = \lambda_2(x,\nu-1)P(G_{\nu-1}^{x_o,x}) - \lambda_2(x,\nu)P(G_\nu^{x_o,x}) \qquad (2.37)$$

for $\nu = 1, 2, \ldots$, and

$$\frac{\partial P(G_o^{x_o,x})}{\partial x} = - \lambda_2(x,o)P(G_o^{x_o,x}) \qquad (2.38)$$

for $\nu = 0$, which finally gives

$$P(G_\nu^{x_o,x}) = \exp\left[-\int_{x_o}^{x}\lambda_2(s,\nu)ds\right]\cdot \int_{x_o}^{x} \lambda_2(x_1,\nu-1)$$

$$\exp\left[\int_{x_o}^{x_1} \Delta\lambda_2(s,\nu)ds\right]\cdot \int_{x_o}^{x_1}\lambda_2(x_2,\nu-2) \exp \int_{x_o}^{x_2} \Delta\lambda_2(s,\nu-1)ds\cdot$$

$$\ldots \int_{x_o}^{x_{\nu-2}} \lambda_2 (x_{\nu-1},1) \exp\left[\int_{x_o}^{x_{\nu-1}} \Delta\lambda_2(s,2)\,ds\right] \cdot \int_{x_o}^{x_{\nu-1}} \lambda_2(x,o)$$

$$\exp\left[\int_{x_o}^{x_\nu} \Delta\lambda_2(s,1)\,ds\right] dx_\nu dx_{\nu-1} \ldots dx_1 , \qquad (2.39)$$

where s, x_1, x_2, ..., x_ν are dummy variables, and

$$\Delta\lambda_2(s,\nu-i) = \lambda_2(s,\nu-i) - \lambda_2(s,\nu-i-1) \ . \qquad (2.40)$$

For the assumption $\lambda_2(t,\nu) \equiv \lambda_2(t)$, eq. (2.39) gives

$$\boxed{P(G_\nu^{x_o,x}) = \exp\left[-\int_{x_o}^{x} \lambda_2(s)\,ds\right] \cdot \frac{\left[\int_{x_o}^{x} \lambda_2(s)\,ds\right]^\nu}{\nu!} \ .} \qquad (2.41)$$

2.5 Some basic properties of functions $\lambda_1(t,\nu)$ and $\lambda_2(x,\nu)$. The distribution functions of stochastic processes η_t, η_x, τ_ν, X_ν and Z_ν are related to probabilities of random events $E_\nu^{t_o,t}$ and $G_\nu^{x_o,x}$ which in turn are functions of $\lambda_1(t,\nu)$ and $\lambda_2(x,\nu)$, respectively. These two functions represent the essence of the entire theory and as such deserve particular attention.

In the following it is assumed

$$\lambda_1(t,\nu) \equiv \lambda_1(t) \ , \quad \lambda_2(x,\nu) \equiv \lambda_2(x) \ . \qquad (2.42)$$

Consider the functions $\Lambda_1(t_o,t)$ and $\Lambda_2(x_o,x)$ that represent mathematical expectations of random variables η_t and η_x, respectively. It is not difficult to prove (see Appendix A) that for any $\Delta t > 0$

$$\Lambda_1(t_o,t+\Delta t) = \Lambda_1(t_o,t) + \Lambda_1(t,t+\Delta t) \ . \qquad (2.43)$$

Similarly, for any $\Delta x > 0$ if follows

$$\Lambda_2(x_o,x+\Delta x) = \Lambda_2(x_o,x) + \Lambda_2(x, x+\Delta x) \ . \qquad (2.44)$$

On the basis of definition, $\Lambda_1(t, t+\Delta t)$ represents the average number of complete storm events in $(t, t+\Delta t]$. On the basis of the nature of the precipitation phenomenon, it looks realistic to assume $\Lambda_1(t, t+\Delta t) \to 0$ if $\Delta t \to 0$. Since

$$\Lambda_1(t, t+\Delta t) = \Lambda_1(t_o,t+\Delta t) - \Lambda_1(t_o,t) \ ,$$

it follows that $\Lambda_1(t_o,t)$ is a continuous function. Now consider the derivatives of expressions (2.26) and (2.27). If one multiplies the left and right side of eq. (2.26) by ν and takes the sum from $\nu = 1$ to $\nu = \infty$, it turns out

$$\sum_{\nu=1}^{\infty} \nu \frac{\partial P(E^{t_o,t})}{\partial t} = - \sum_{\nu=1}^{\infty} \nu\lambda_1(t,\nu)P(E_\nu^{t_o,t})$$

$$+ \sum_{\nu=1}^{\infty} \nu\lambda_1(t,\nu-1)P(E_{\nu-1}^{t_o,t}) = \sum_{\nu=o}^{\infty} \lambda_1(t,\nu)P(E_\nu^{t_o,t})$$

i.e.,

$$\frac{\partial\Lambda_1(t_o,t)}{\partial t} = \sum_{\nu=o}^{\infty} \lambda_1(t,\nu)P(E_\nu^{t_o,t}) \ . \qquad (2.45)$$

On the other hand, from the relation of (2.15) results

$$P(E_1^{t,t+\Delta t} \cap E_\nu^{t_o,t}) = \lambda_1(t,\nu)P(E_\nu^{t_o,t})\Delta t + P(E_\nu^{t_o,t}) + o(\Delta t)$$

or

$$P(E_1^{t,t+\Delta t}) = \sum_{\nu=o}^{\infty} \lambda_1(t,\nu)P(E_\nu^{t_o,t})\Delta t + o(\Delta t) \ .$$

Therefore, it follows

$$\lambda_1(t) = \lim_{\Delta t \to o} \frac{P(E_1^{t,t+\Delta t})}{\Delta t} = \sum_{\nu=o}^{\infty} \lambda_1(t,\nu)P(E_\nu^{t_o,t}) \ , \qquad (2.46)$$

and by virtue of (2.45) and (2.46) it turns out

$$\frac{\partial\Lambda_1(t_o,t)}{\partial t} = \lambda_1(t) \ . \qquad (2.47)$$

On the basis of eq. (2.32), the density function of (2.22) becomes

$$\boxed{f_\nu(t) = \frac{\lambda_1(t)}{\Gamma(\nu)} e^{-\int_{t_o}^{t} \lambda_1(s)\,ds} \left[\int_{t_o}^{t} \lambda_1(s)\,ds\right]^{\nu-1}} \qquad (2.48)$$

where $\nu = 1, 2, \ldots$.

Another useful interpretation of $\lambda_1(t)$ is in considering the expectation of η_t, (using eq. (2.32))

$$E(\eta_t) = \lambda_1(t_o,t) = \sum_{\nu=1}^{\infty} \nu e^{-\int_{t_o}^{t} \lambda_1(s)\,ds} \frac{\left[\int_{t_o}^{t} \lambda_1(s)\,ds\right]^\nu}{\nu!} =$$

$$= \int_{t_o}^{t} \lambda_1(s)\,ds$$

so the average number of storms in the interval $(t_o,t]$ is equal to

$$E(\eta_t) = \lambda_1(t_o,t) = \int_{t_o}^{t} \lambda_1(s)\,ds \ ,$$

in which $\lambda_1(t)$ represents a kind of density of storms in a unit of time.

Because of daily and seasonal variations, it is realistic to expect that $\lambda_1(t)$ is a periodic time function, with a day or a year as periods, individual or superposed cycles.

If the interval of time $(t_o, t]$ is sufficiently small or if the ξ_t-process is stationary, so that the function $\lambda_1(t)$ is approximately equal to a constant λ_1, and t_o may be taken as zero, then the number of storms η_t is distributed according to Poisson distribution as

$$\boxed{P\{\eta_t = \nu\} = P(E_\nu^{t_o,t}) = e^{-\lambda_1 t} \cdot \frac{(\lambda_1 t)^\nu}{\nu!} \quad .} \tag{2.49}$$

Similarly, assuming $\Lambda_2(x_o,x)$ is a differentiable function, and

$$\lambda_2(x,\nu) \equiv \lambda_2(x)$$

it follows that

$$P(G_\nu^{x_o,x}) = e^{-\int_{x_o}^{x} \lambda_2(s)ds} \cdot \frac{\left[\int_{x_o}^{x} \lambda_2(s)ds\right]^\nu}{\nu!} \,, \tag{2.50}$$

and

$$E(\eta_x) = \Lambda_2(x_o,x) = e^{-\int_{x_o}^{x} \lambda_2(s)ds} \sum_{\nu=1}^{\infty} \nu \frac{\left[\int_{x_o}^{x} \lambda_2(s)ds\right]^\nu}{\nu!}$$

$$= \int_{x_o}^{x} \lambda_2(s)ds \ . \tag{2.51}$$

Therefore, by virtue of (2.50) the density function of (2.24) of X_ν becomes

$$\boxed{a_\nu(x) = \frac{\lambda_2(x)}{\Gamma(\nu)}\, e^{-\int_{x_o}^{x} \lambda_2(s)ds} \left[\int_{x_o}^{x} \lambda_2(s)ds\right]^{\nu-1} .} \tag{2.52}$$

With respect to seasonal variations it is realistic to expect the function $\lambda_1(t)$ is a periodic function with a year as the period (See Fig. 4). In fact, this assertion can be mathematically proved. Toward this end consider stochastic processes ξ_t and $\eta_{t_o,t}$; since ξ_t has the periodic probability structure it is obvious that

$$P\{\eta_{t_o,t} = \nu\} = P\{\eta_{t_o+\tau} = \nu_{t+\tau}\} \tag{2.53}$$

where τ is the period. For example, the probability that in November of one year the ν storms will occur is equal to the probability of the same number ν of storms to occur in anyone of other years. Therefore,

$$P(E_\nu^{t_o,t}) = P(E_\nu^{t_o+\tau,\ t+\tau}) \ ,$$

i.e.,

$$e^{-\int_{t_o}^{t} \lambda_1(s)ds} \frac{\left[\int_{t_o}^{t} \lambda_1(s)ds\right]^\nu}{\nu!} = e^{-\int_{t_o+\tau}^{t+\tau} \lambda_1(s)ds} \frac{\left[\int_{t_o+\tau}^{t+\tau} \lambda_1(s)ds\right]^\nu}{\nu!} .$$

Using the substitution

$$u + \tau = s \qquad ds = du,$$

the left side of the last equation becomes

$$e^{-\int_{t_o+\tau}^{t+\tau} \lambda_1(s)ds} \frac{\left[\int_{t_o+\tau}^{t_o+\tau} \lambda_1(s)ds\right]^\nu}{\nu!}$$

$$\equiv e^{-\int_{t_o}^{t} \lambda_1(u+\tau)du} \frac{\left[\int_{t_o}^{t} \lambda_1(u+\tau)du\right]^\nu}{\nu!}$$

or

$$e^{-\int_{t_o}^{t} \lambda_1(s)ds} \frac{\left[\int_{t_o}^{t} \lambda_1(s)ds\right]^\nu}{\nu!} \equiv e^{-\int_{t_o}^{t} \lambda_1(s+\tau)ds} \frac{\left[\int_{t_o}^{t} \lambda_1(s+\tau)ds\right]^\nu}{\nu!}$$

which is possible only if

$$\lambda_1(s) = \lambda_1(s+\tau) \ ,$$

which proves that $\lambda_1(t)$ is a periodic function (see Fig. 4).

On the basis of eq. (2.51), the average number of storms needed to produce amounts of precipitation $(x-x_o)$ is equal to $\int_{x_o}^{x} \lambda_2(s)ds$. Therefore, the function $\lambda_2(x)$ is a characteristic of the amount of precipitation of one storm. It should be a constant or periodic function.

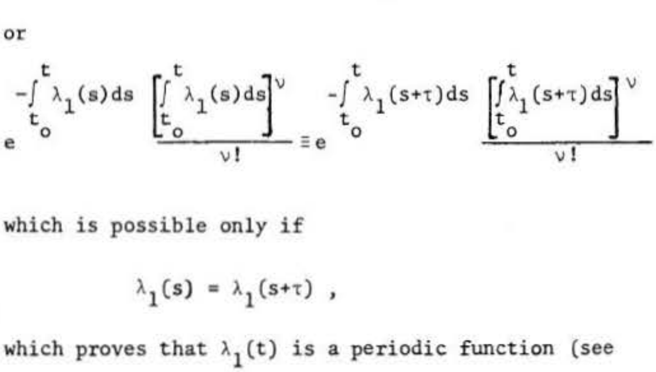

Fig. 4 Graphical presentation of the function $\lambda_1(t)$

For a small amount of precipitation (say $x-x_o$), or if the ξ_t-process is stationary, $\lambda_2(x)$ is approximately equal to a constant λ_2, and x_o may be taken as zero, so the probability of the number of storms for the given amount of precipitation x, according to eq. (2.50), becomes a Poissonian distribution ($x_o = 0$) and

$$P\{\eta_x = \nu\} = P(G_\nu^{x_o, x}) = e^{-\lambda_2 x} \cdot \frac{(\lambda_2 x)^\nu}{\nu} . \qquad (2.54)$$

The mathematical expectation of the variable τ_ν can be obtained from eq. (2.48), or from its density function, as

$$E(\tau_\nu) = \frac{1}{\Gamma(\nu)} \int_{t_o}^{\infty} t\lambda_1(t)\exp[-\int_{t_o}^{t} \lambda_1(s)ds]\{\int_{t_o}^{t} \lambda_1(s)ds\}^{\nu-1}dt.$$

In a similar way

$$E(X_\nu) = \frac{1}{\Gamma(\nu)} \int_{x_o}^{\infty} x\lambda_2(x)\exp[-\int_{x_o}^{x} \lambda_2(s)ds]\{\int_{x_o}^{x} \lambda_2(s)ds\}^{\nu-1}dx.$$

The estimate of these integrals can be made under the assumption that both $\lambda_1(t)$ and $\lambda_2(x)$ are bounded functions, with $\lambda_1(t) > 0$ and $\lambda_2(x) > 0$ for all t and x respectively. In other words, one can prove

$$\frac{\nu}{\overline{\lambda}_1} \le E(\tau_\nu) \le \frac{\nu}{\underline{\lambda}_1} \qquad \frac{\nu}{\overline{\lambda}_2} \le E(X_\nu) \le \frac{\nu}{\underline{\lambda}_2} , \qquad (2.55)$$

where $\underline{\lambda}_1 = \inf \lambda_1(t)$ is the lower bound of $\lambda_1(t)$ for any t, $\overline{\lambda}_1 = \sup \lambda_1(t)$, the upper bound of $\lambda_1(t)$; and $\underline{\lambda}_2 = \inf \lambda_2(x)$, $\overline{\lambda}_2 = \sup \lambda_2(x)$.

It can be shown that (see Appendix B)

$$\frac{1}{\overline{\lambda}_1} \le E(\tau_\nu) - E(\tau_{\nu-1}) \le \frac{1}{\underline{\lambda}_1} \qquad (2.56)$$

and

$$\sum_{\nu=1}^{\nu} \frac{1}{\overline{\lambda}_1} \le \sum_{\nu=1}^{\nu} \{E(\tau_\nu) - E(\tau_{\nu-1})\} \le \sum_{\nu=1}^{\nu} \frac{1}{\underline{\lambda}_1} , \qquad (2.57)$$

and assuming $E(\tau_o) = 0$, then

$$\frac{\nu}{\underline{\lambda}_1} \le E(\tau_\nu) \le \frac{\nu}{\overline{\lambda}_1} . \qquad (2.58)$$

In a similar way one can show

$$\frac{\nu}{\overline{\lambda}_2} \le E(X_\nu) \le \frac{\nu}{\underline{\lambda}_2} \qquad (2.59)$$

where

$$\overline{\lambda}_2 = \sup \lambda_2(x) \quad \text{and} \quad \underline{\lambda}_2 = \inf \lambda_2(x) ,$$

assuming $\underline{\lambda}_2 > 0$.

2.6. Some asymptotical distributions. In the previous sections the density functions $f_\nu(t)$ and $a_\nu(x)$ of the random variables τ_ν and X_ν have been theoretically derived for all $\nu = 1, 2, \ldots$. It was shown that under the relations of eqs. (2.13) and (2.14)

$$f_\nu(t) = \frac{\lambda_1(t)}{\Gamma(\nu)} e^{\int_o^t \lambda_1(s)ds} (\int_o^t \lambda_1(s)ds)^{\nu-1} \qquad (2.60)$$

$$a_\nu(x) = \frac{\lambda_2(x)}{\Gamma(\nu)} e^{-\int_o^x \lambda_2(s)ds} (\int_o^x \lambda_2(s)ds)^{\nu-1} , \qquad (2.61)$$

where $\nu = 1, 2, \ldots$ and $\Gamma(\nu) = (\nu-1)!$

Now, an effort is made to investigate behavior of the functions of (2.60) and (2.61) for large values of ν. It will be proven that both of the density functions tend to the Gaussian distribution if $\nu \to \infty$, if periodic functions $\lambda_1(t)$ and $\lambda_2(x)$ are bounded. In other words, it can be shown that for large ν

$$f_\nu(t)dt \simeq \frac{\lambda_1^*}{\sqrt{2\pi(\nu-1)}} e^{\frac{(t - \nu/\lambda_1^*)^2}{2(\nu-1)(\lambda_1^*)^{-2}}} dt \ ; \underline{\lambda}_1 < \lambda_1^* < \overline{\lambda}_1 \qquad (2.62)$$

$$a_\nu(x)dx \simeq \frac{\lambda_2^*}{\sqrt{2\pi(\nu-1)}} e^{\frac{(x - \nu/\lambda_2^*)^2}{2(\nu-1)(\lambda_2^*)^{-2}}} dx \ ; \underline{\lambda}_2 < \lambda_2^* < \overline{\lambda}_2 . \qquad (2.63)$$

Indeed, consider first the density function of (2.60); since, according to hypothesis, $\lambda_1(t)$ is periodic and bounded, then for large t approximately

$$\int_o^t \lambda_1(s)ds \simeq \lambda_1^* t . \qquad (2.64)$$

Therefore, on the basis of eq. (2.64) and the Stirling's formula

$$n! \simeq n^n e^{-n} \sqrt{2\pi n} ,$$

eq. (2.60) becomes

$$f_\nu(t)dt \simeq \frac{\lambda_1^*}{\sqrt{2\pi(\nu-1)}} e^{-\lambda_1^* t} \frac{\lambda_1^* t}{\nu-1}^{\nu-1} e^{\nu-1} dt$$

$$\simeq \frac{\lambda_1^*}{\sqrt{2\pi(\nu-1)}} e^{-(\lambda_1^* t-\nu+1)} (\frac{\lambda_1 t}{\nu-1})^{\nu-1} dt .$$

By virtue of the substitution

$$u = \lambda_1^* t - \nu+1 ; \ dt = \frac{du}{\lambda_1^*} . \qquad (2.65)$$

Then

$$f_\nu(t)dt \simeq \frac{1}{\sqrt{2\pi(\nu-1)}}\, e^{-u}\, (1+\frac{u}{\nu-1})^{\nu-1}\, du \ .$$

If ν is a very large number it follows that

$$(\nu-1)\ln(1+\frac{u}{\nu-1}) \simeq (\nu-1)\ \{\frac{u}{\nu-1} - \frac{u^2}{2(\nu-1)^2}\} = u - \frac{u^2}{2(\nu-1)}$$

or

$$(1+\frac{u}{\nu-1})^{\nu-1} \simeq e^{u} - \frac{u^2}{2(\ -1)} \ ;$$

therefore,

$$f_\nu(t)dt \simeq \frac{1}{\sqrt{2\pi(\nu-1)}}\, e^{-\frac{u^2}{2(\nu-1)}}\, du \ .$$

Finally, by virtue of eq. (2.65)

$$f_\nu(t)dt \simeq \frac{\lambda_1^*}{\sqrt{2\pi(\nu-1)}}\, e^{\frac{(t-\frac{\nu-1}{\lambda_1^*})}{2(\nu-1)(\lambda^*)^{-2}}}\, dt \ ,$$

i.e., for very large ν, $f_\nu(t)$ tends to the Gaussian distribution with mean and variance equal to

$$\frac{\nu-1}{\lambda_1^*} \quad \text{and} \quad \frac{\nu-1}{(\lambda_1^*)^2} \ , \tag{2.66}$$

respectively.

In a similar way one can conclude eq. (2.63) is correct as well. It is easy to prove that the result is not a trivial consequence of the central limit theorem, regardless of the fact X_ν is the sum of random variables, i.e.,

$$X_\nu = \sum_{k=1}^{\nu} Z_\nu \ ,$$

because none of the conditions of the central limit theorems are satisfied.

2.7 The distribution of X_t. The random variable X_t was defined as the total amount of precipitation during the interval $(t_o,t]$. To determine the distribution of X_t, consider the event

$$B_t(x) = \{\omega; X_t \le x\}, \quad x \ge x_o \ .$$

The corresponding probability $P[B_t(x)]$ represents the one-dimensional distribution function $F_t(x)$ of the stochastic process X_t, i.e.,

$$P\{\omega;\ X_t \le x\} = F_t(x) \ .$$

On the basis of eq. (2.10), the event $B_t(x)$ can also be written in the form

$$B_t(x) = \bigcup_{\nu=o}^{\infty} [E_\nu^{t_o,t} \cap B_t(x)] \ , \tag{2.67}$$

which implies

$$F_t(x) = \sum_{\nu=o}^{\infty} P[E_\nu^{t_o,t} \cap B_t(x)] \ , \tag{2.68}$$

In Appendix C, it is shown the above equation can be reduced to:

$$F_t(x) = \sum_{\nu=o}^{\infty} \sum_{j=\nu}^{\infty} P(E_\nu^{t_o,t} \cap G_j^{x_o,x}) \ . \tag{2.69}$$

If it is assumed that the interval $(t_o,t]$ is sufficiently small, then the events $E_\nu^{t_o,t}$ and $G_j^{x_o,x}$ are indepenent. The probabilities of these events can be represented by the Poisson distribution (thereby assuming constant λ_1 and λ_2). The distribution function of X_t can be written as:

$$F_t(x) = \sum_{\nu=o}^{\infty} \sum_{j=\nu}^{\infty} e^{-(\lambda_1 t+\lambda_2 x)} \frac{(\lambda_1 t)^\nu (\lambda_2 x)^j}{\nu!\ \ j!} \ . \tag{2.70}$$

For $x = o$, eq. (2.70) reduces to

$$F_t(o) = e^{-\lambda_1 t} \ .$$

The density function of the random variable X_t can be found by differentiation of $F_t(x)$:

$$f^*(x) = \frac{\partial F_t(x)}{\partial x}$$

$$= e^{-\lambda_1 t} \sum_{\nu=o}^{\infty} \frac{(\lambda_1 t)^\nu}{\nu!} [-\lambda_2 e^{-\lambda_2 x} \sum_{j=\nu}^{\infty} \frac{(\lambda_2 x)^j}{j!}$$

$$+ e^{-\lambda_2 x} \sum_{j=\nu}^{\infty} \frac{2(\lambda_2 x)^{j-1}}{(j-1)!}] = \lambda_2 e^{-\lambda_1 t-\lambda_2 x} \sum_{\nu=1}^{\infty} \frac{(\lambda_2 x)^{\nu-1}}{(\nu-1)!} \frac{(\lambda_1 t)^\nu}{\nu!}$$

$$= \lambda_2 e^{-\lambda_1 t-\lambda_2 x} \sum_{k=o}^{\infty} \frac{(\lambda_1 t)^{k+1}}{(k+1)!} \frac{(\lambda_2 x)^k}{k!}$$

$$= \lambda_2 \sqrt{\frac{\lambda_1 t}{\lambda_2 x}}\, e^{-\lambda_1 t-\lambda_2 x} \sum_{k=o}^{\infty} \frac{(2\sqrt{\lambda_1\lambda_2 tx})^{2k+1}}{2^{2k+1}(k+1)!k!}$$

$$= \lambda_2 \sqrt{\frac{\lambda_1 t}{\lambda_2 x}}\, e^{-\lambda_1 t-\lambda_2 x}\, I_1(2\sqrt{\lambda_1\lambda_2 tx}) \ . \tag{2.71}$$

Here, $I_1(2\sqrt{\lambda_1\lambda_2 tx})$ is the modified Bessel function of the first order. To facilitate determination of the value of the modified Bessel functions for large arguments from tables (British Association Mathematical Tables, 1958), eq. (2.70) can also be written as

$$\boxed{f_t^*(x) = \lambda_2 e^{-(\sqrt{\lambda_1 t} - \sqrt{\lambda_2 x})} \sqrt{\frac{\lambda_1 t}{\lambda_2 x}}\, e^{-2\sqrt{\lambda_1 \lambda_2 t x}}\, I_1(2\sqrt{\lambda_1 \lambda_2 t x})} \tag{2.72}$$

However, because of the discontinuity of the distribution function at $x = o$, for the density function as shown above

$$\int_o^\infty f_t^*(x)\,dx = 1 - e^{-\lambda_1 t}$$

The density function therefore must be corrected, by using the Dirac δ function and the knowledge that

$$F_t(o) = e^{-\lambda_1 t} \; ,$$

to read as follows

$$\boxed{f_t(x) = \delta(x) e^{-\lambda_1 t} + f_t^*(x)} \tag{2.73}$$

where

$$\delta(x) = o \quad \text{for } x \neq o$$

$$\delta(x) = \infty \quad \text{for } x = o,$$

and

$$\int_{-\infty}^{+\infty} \delta(x)\,dx = 1 \; .$$

The mathematical expectation of X_t is equal to

$$\boxed{E(X_t) = \frac{\lambda_1}{\lambda_2} t} \tag{2.74}$$

(See Appendix C).

The variance is equal to

$$\boxed{\mathrm{var}(X_t) = \frac{2\lambda_1 t}{\lambda_2^2}} \tag{2.75}$$

Chapter III

RESEARCH DATA ASSEMBLY FOR TESTING MODELS

3.1 Objectives of data assembly for investigation of stochastic process of precipitation. The theoretical background, as given in the previous chapter, shows that some properties of periodic-probability structure of stochastic process $\{\xi_t\}$ have been expressed in terms of two parameters which change along the year: λ_1, which is the density of number of storms in time, and λ_2, which is the yield characteristic of storms, in time. Distributions of all random variables, which are given as functions of $\{\xi_t\}$ and were studied in the previous chapter, are expressed in terms of either λ_1 or λ_2 or of both.

To verify how well the theoretical distributions of random variables, which are functions of the $\{\xi_t\}$-process, fit distributions of observed samples of various precipitation storm data, the research data are first assembled. Then, parameters λ_1 and λ_2 are estimated as they change throughout the year. Finally, empirical distributions are compared with theoretical probability distributions, the latter based on the computed λ_1- and λ_2-parameters. The first objective of the research data assembly and analysis is to show approximate variations of λ_1 and λ_2 through the year for some precipitation stations, and particularly whether they are or are not periodic in nature. Then, by using these values of λ_1 and λ_2, theoretical probability distributions of new variables are derived and compared with the corresponding distributions as obtained from data.

3.2 Selection of number of precipitation stations. Precipitation data are currently available in the form of discrete time series of hourly, daily, or monthly values. Rarely are the continuous series of precipitation intensities available. The monthly-interval series is not relevant to the present study. Series of hourly precipitation contain numerous data. They are often cumbersome to handle, even by the computer tape input. The use of time series of daily precipitation is imposed by the type of data available.

The approach in this study is to show, by a few examples, how the theory may be applied to particular cases, rather than to use an exhaustive approach of applying the theory to hundreds of stations for obtaining statistical characteristics of goodness of fit of theoretical probability distributions to empirical distributions. Therefore, only three precipitation stations of daily data and a station of hourly data are chosen as the research material for objectives of this study. An exhaustive approach will be feasible once a methodology of application of the above theory is well designed and tested on a small number of representative examples.

The three stations of daily values are:

(1) Durango, Colorado, for 71 years of observations, 1895 to 1965, on the western slopes of the Rocky Mountains, with an average annual precipitation of P = 19.02 inches;

(2) Fort Collins, Colorado, for 69 years of observations, 1898 to 1966, on the eastern slopes of the Rocky Mountains, with an average annual precipitation of P = 14.32 inches; and

(3) Austin, Texas, for 70 years of observations, 1898 to 1967, which is influenced by the Gulf of Mexico air masses and clearly has two rainfall seasons, with an average annual precipitation of P = 33.02 inches. The records of this station 35 years prior to 1898 have missing data, from time to time, and were not found feasible for computer oriented processing.

Hourly values are assembled only for a station at Ames, Iowa, for a period of 18 years, January 1949 through December 1966. The average annual precipitation of this station for this period is P = 28.89 inches.

3.3 Basic characteristics of daily and hourly precipitation series. In order to study basic properties of storms by daily or hourly precipitation data, a year is divided into 28 intervals, each 13 days long, for a total of 364 days. The 365th day is neglected, or in case the year has 366 days, the data of the two last days are neglected, whenever the properties of 13-day intervals are studied. The only reason for selecting 13-day intervals was the fact that 13 x 28 = 364 so that intervals are equal in size, as opposed to months which are of different size, and that the residuals (loss of information) are only 1 or 2 days per year. Any other interval could be similarly used. It was also considered in this study that 28 values of 13-day intervals will give a much better picture of how λ_1 and λ_2 vary within the year. A much greater number would increase the sampling variations in λ_1 and λ_2 while a smaller number of intervals may decrease the information about them.

Basic characteristics of data are defined in this study as the means, standard deviations, and coefficients of variation of individual 13-day intervals. If n = the number of years of data, then

$$m_\tau = \frac{1}{n} \sum_{p=1}^{n} x_{p,\tau} \tag{3.1}$$

are these means, with $x_{p,\tau}$ the total precipitation of an interval for a given year, with intervals $\tau = 1,2,\ldots,28$ and years $p = 1,2,\ldots,n$. Similarly, the 28 standard deviations are

$$s_\tau = [\frac{1}{n} \sum_{p=1}^{n} (x_{p,\tau} - m_\tau)^2]^{\frac{1}{2}} \tag{3.2}$$

and the corresponding coefficients of variation are

$$_\tau C_v = \frac{s_\tau}{m_\tau} \tag{3.3}$$

with s_τ given by eq. (3.2) and m_τ by eq. (3.1). If each value of eqs. (3.1) and (3.2) is divided by 13, then they give the densities of daily precipitation characteristics of means and standard deviations, over 13-day intervals.

Figures 3.1 through 3.3 give densities of daily precipitation characteristics referring to 13-day intervals, for the means (in inches per day), standard deviations (in inches per day), and coefficients of variation for precipitation series of Durango (Colorado), Fort Collins (Colorado) and Austin (Texas), respectively. Similarly, Fig. 3.4 gives these densities (this time over hours) and coefficients of variation for Ames (Iowa). Left hand graphs in these figures show the computed densities of means and the fitted periodic components of significant harmonics in these densities. The averages of these graphs multiplied by 365 give the total average annual precipitation of each station. Central graphs show the computed densities of standard deviations and either the fitted periodic components of significant harmonics or the mean values of these densities of standard deviations. To obtain the standard deviation for a 13-day interval, the values of central graphs must be multiplied by 13 for daily precipitation, or by 13 x 24 = 312 for hourly

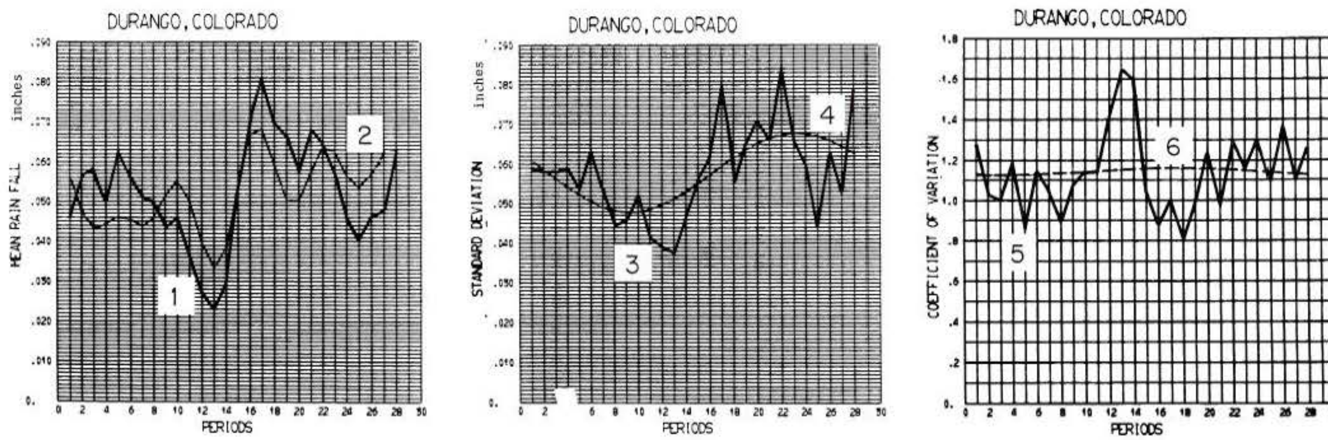

Fig. 3.1 Densities of means and standard deviations, and the coefficients of variation of 28 intervals of the year, each 13-days long, for daily precipitation series at Durango, Colorado (1895 - 1965, for n = 71 years): (1) computed densities of means, m_τ, in inches per day; (2) fitted periodic component to densities of means, μ_τ, composed of significant harmonics; (3) computed densities of standard deviations, s_τ, in inches per day; (4) fitted periodic component to densities of standard deviations, σ_τ, composed of only the first significant harmonic; (5) computed coefficients of variation, ${}_\tau C_v$, and (6) fitted periodic component, β_τ, to coefficients of variation, composed of only the first significant harmonic.

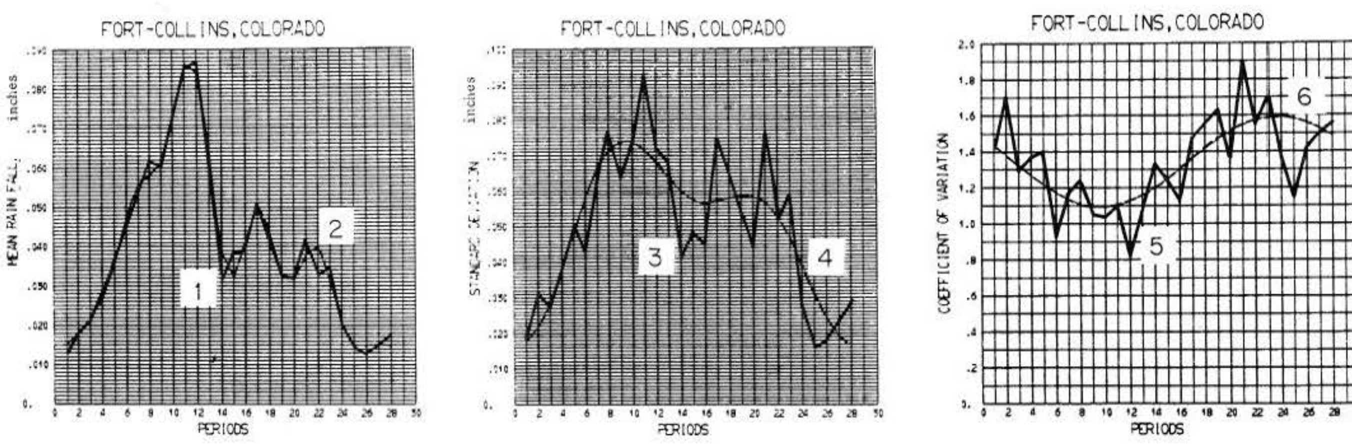

Fig. 3.2 Densities of means and standard deviations, and the coefficients of variation of 28 intervals of the year, each 13-days long, for daily precpipitation series at Fort Collins, Colorado, (1898 - 1966, or n = 69 years): (1) computed densities of means, m_τ, in inches per day; (2) fitted periodic component to densities of means, μ_τ, composed of significant harmonics; (3) computed densities of standard deviations, s_τ, in inches per day; (4) fitted periodic component to densities of standard deviations, σ_τ, composed of significant harmonics; (5) computed coefficients of variation, ${}_\tau C_v$, and (6) fitted periodic component to coefficients of variation, β_τ, composed of only the first significant harmonic.

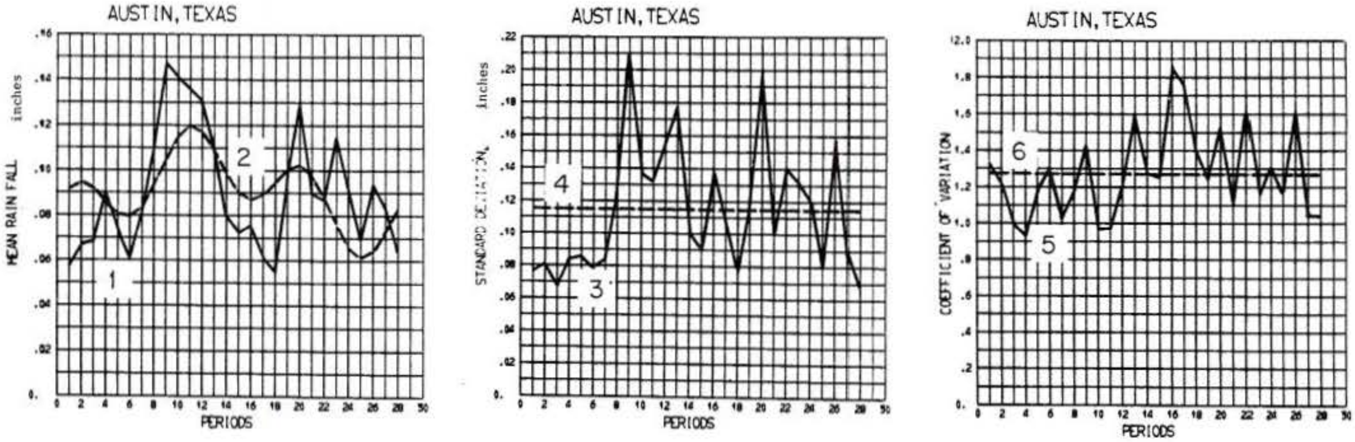

Fig. 3.3 Densities of means and standard deviations, and the coefficients of variation of 28 intervals of the year, each 13-days long, for daily precipitation series at Austin, Texas (1898 - 1967, or 70 years): (1) computed densities of means, m_τ, in inches per day; (2) fitted periodic component to densities of means, μ_τ, composed of significant harmonics; (3) computed densities of standard deviations, σ_τ, in inches per day; (4) the mean of 28 values of computed s_τ, with no harmonic being significant; (5) computed coefficients of variation, ${}_\tau C_v$, and (6) the mean of 28 values of computed ${}_\tau C_v$, with no harmonic being significant.

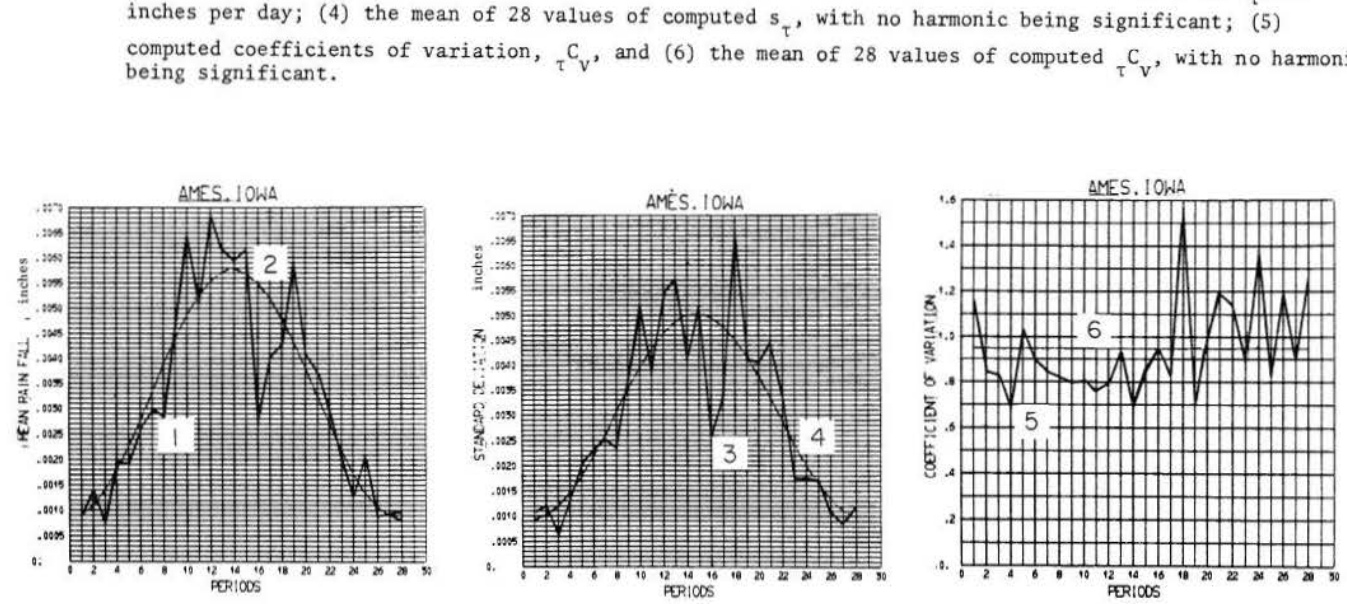

Fig. 3.4 Densities of means and standard deviations, and the coefficients of variation of 28 intervals of the year, each 13-days long, for hourly precipitation series at Ames, Iowa (1949 - 1966, or 18 years): (1) computed densities of means, m_τ, in inches per hour; (2) fitted periodic component, μ_τ, to densities of means, composed of the significant first harmonic; (3) computed densities of standard deviations, s_τ, in inches per hour; (4) fitted periodic component, σ_τ, to densities of standard deviations, composed of only the first significant harmonic; (5) computed coefficients of variation, ${}_\tau C_v$, and (6) the mean of 28 values of computed ${}_\tau C_v$, with no harmonic significant.

precipitation. Righthand graphs show the computed coefficients of variation and either the fitted periodic components of significant harmonics or the mean value of these coefficients.

Significant harmonics are determined in this case by using the following approach. Any parameter ν_τ has the equation for its periodic component:

$$\nu_\tau = \nu_x + \sum_{j=1}^{m} (A_j \cos \lambda_j \tau + B_j \sin \lambda_j \tau), \qquad (3.4)$$

where $\lambda_j = 2\pi j/\omega$ is the angular frequency, ω = the number of ordinates in the basic period in ν_τ, m = the total number of harmonics inferred as significant in the Fourier series mathematical description of the periodic component, A_j and B_j are Fourier coefficients, j = significant harmonics, and ν_x = the mean of ω values of ν_τ with $\tau = 1,2,\ldots,\omega$. In this particular case ω = 28, and ν_τ refers to either μ_τ, σ_τ or β_τ, respectively for m_τ, s_τ, and ${}_\tau C_v$.

Coefficients A_j and B_j of eq. (3.4) are estimated by

$$A_j = \frac{2}{\omega} \sum_{\tau=1}^{\omega} (\nu_\tau - \nu_x) \cos \frac{2\pi j \tau}{\omega} \tag{3.5}$$

and

$$B_j = \frac{2}{\omega} \sum_{\tau=1}^{\omega} (\nu_\tau - \nu_x) \sin \frac{2\pi j \tau}{\omega} \tag{3.6}$$

For $\omega = 28$, the maximum number of harmonics is $\omega/2$, or $\omega/2 = 14$. Experience shows that the first six harmonics are the most important, or $1 \leq j \leq 6$; harmonics beyond the sixth are very rarely shown to be significant.

The square of amplitude of any harmonic is

$$C_j^2 = A_j^2 + B_j^2 \tag{3.7}$$

and the part of variation of any parameter ν_τ resulting from this harmonic is $C_j^2/2$. The variance of ν_τ is

$$\text{var } \nu_\tau = \frac{1}{\omega} \sum_{\tau=1}^{\omega} (\nu_\tau - \nu_x)^2, \tag{3.8}$$

and the parameter

$$g = \frac{C_{max}^2}{\sum_{j=1}^{\omega/2} C_j^2} \tag{3.9}$$

is Fisher's parameter for testing the significance of the harmonic with the largest C_j. As only six harmonics are computed in this study for the three parameters, then

$$g = \frac{C_{max}^2}{2 \text{ var } \nu_\tau} \tag{3.10}$$

replaces eq. (3.9) because both give identical values of g. By using R.A. Fisher's [2] expressions and tables, and probability P = 5%, the critical value g_c is determined for testing the significance of g of eq. 3.10. If $g > g_c = 0.3517$, the harmonic is considered as significant. For the second, third, ... highest values of C_j^2, then

$$g = \frac{C_j^2}{2 \text{ var } \nu_\tau - \sum_{i=1}^{j-1} C_i^2} \tag{3.11}$$

where ΣC_i^2 represents all harmonics with C_i greater than C_j, with i denoting the sequence of C_i^2 from the highest to the smallest C_j. The same test is repeated for g of eq. (3.11) as for g of eq. (3.10).

Figures 3.1 through 3.4 show the fitted periodic components composed of harmonics which have been found significant by the above procedure. It is expected differences between computed values of m_τ, s_τ and ${}_\tau C_v$ and the fitted periodic components of significant harmonics, μ_τ, σ_τ and β_τ, represent only the sampling random variation about these periodic components.

Figure 3.1 for Durango Precipitation Station shows clearly a periodicity in both the means and standard deviations of 28 intervals of 13-days. These periodicities in m_τ and s_τ are similar, though the sampling variations somewhat mask this parallelism. The significant harmonics of 365 days of the periodicity in the coefficient of variation has a very small amplitude. For practical purposes, it can be neglected, with ${}_\tau C_v$ being approximately a constant.

Figure 3.2 for Fort Collins Precipitation Station shows a very pronounced periodicity in all three parameters, m_τ, s_τ and ${}_\tau C_v$, though the periodicity in ${}_\tau C_v$ has a much smaller amplitude of the basic 365-day harmonic. A comparison between Durango and Fort Collins stations points out some clear differences in time patterns of precipitation amounts in the sequence of 28 values of 13-day intervals.

Figure 3.3 for Austin Precipitation Station shows a periodicity in m_τ, while harmonics in s_τ are not shown as being significant. The coefficient of variation, ${}_\tau C_v$, does not show significant harmonics either. A parallelism between m_τ and s_τ series is evident, though the sampling variations make this property less obvious.

Figure 3.4 for Ames Precipitation Station shows a periodicity in the mean, m_τ, and the standard deviations, s_τ. The parallelism of periodic components is a striking feature of this series of hourly precipitation. Because of a short period of data of only 18 years, the variations of observed values m_τ and s_τ about the fitted periodic components of μ_τ and σ_τ are likely to be predominately of a sampling character. This aspect is best represented by ${}_\tau C_v$, which does not show any significant harmonic. It may be assumed to be approximately a constant, independent of the time of the year.

3.4 Two approaches of using the research data. The theory in the preceeding chapter is primarily related to precipitation storms as continuous processes whenever it rains. When the precipitation data are given as daily values, the appropriate definition of the storm is required. In that case, and for the purposes of this study, the storm is defined as a precipitation sequence consisting of an uninterrupted number of rainy days. If the records show only one rainy day with the preceeding day and the following day as non-rainy days, then the storm is of 1-day duration and its total amount is the rainfall of that day. If the records show four uninterrupted rainy days, then the storm duration is four days and its total amount is the sum of rainfall for these four days. This definition is, therefore, related to a discrete series. The random variable, which represents the time a storm ends, is now the last day of an uninterrupted sequence of daily rainfall. The storm duration is the number of rainy days of a storm.

Regardless of this storm definition, two approaches were used in processing the research data: (1) Each rainy day is treated as an individual storm event, whether or not it is preceeded or followed by a rainy or non-rainy day, and (2) Each storm is identified as an uninterrupted sequence of rainy days, as defined in the previous paragraph. It is not expected that theoretical development of the preceeding chapter would fit the properties of daily precipitation in the first approach. The second approach is considered more crucial in testing the coincidence of theoretical models to the empirical distributions of random functions investigated in Chapter II. However, the real storm duration may be shorter than the number of uninterrupted sequence of rainy days.

The hourly rainfall of the Ames Precipitation Station is also treated by these two distinct approaches because it is attractive to consider each hourly value as an individual storm event, regardless whether or not it is preceeded or followed by a rainy or non-rainy hour. Second, the storm is defined in this case as an uninterrupted sequence of hourly precipitation greater than zero, preceeded and succeeded by non-rainy hour or hours. Because the thunderstorm type of rainfall may frequently be composed of a succession of rainy and non-rainy hours. The use of each hour of rainfall as being a storm for the study of precipitation phenomenon produces a large number of small storms. For this reason, only careful inference about storms in the first approach, in case of hourly precipitation can be drawn.

In conclusion, the fact that the rainfall data is currently available in the form of hourly or daily values, and not as a continuous $\{\xi_t\}$ random process whenever it rains, makes it difficult for an exact comparison between the theoretical and empirical distributions to be made. Regardless of this, the second approach to both daily and hourly series of storm definitions comes as close to the random process $\{\xi_t\}$ as practically feasible, (although it is somewhat biased).

Chapter IV

DENSITY OF STORMS IN TIME

4.1 Significance and computation of the parameter of density of storms in time. The analysis of six random variables, which are discussed in Chapter II as functions of the stochastic process $\{\xi_t\}$ has shown that some of their distributions are dependent on λ_1. This parameter λ_1 is the average number of storms in a unit of time during any time of the year. It is briefly called in the title of this chapter and in the following text as "the density of storms in time." The basic conclusion in the previous analysis is that the existence of periodicity in the $\{\xi_t\}$-process is reflected as the periodicity in the parameter λ_1. The year is the basic period. In other words, the annual periodicity in the mean, in the standard deviation and eventually in other parameters of the basic process $\{\xi_t\}$ imposes the periodicity in λ_1-parameter. Through λ_1, all other random variables, which are functions of $\{\xi_t\}$-process and dependent on λ_1, should exhibit a periodicity similar to the parameter λ_1.

Numerical characteristics of λ_1-parameter are investigated in this chapter for the four examples of precipitation storm data described in Chapter III. The number of ends of storm events in each 13-day time interval is determined as integers 0, 1, 2, ... , which represent also the number of storms. The total of n values of this number for n years of data is obtained for each 13-day interval and for each station. Then the expected value, the variance and the ratio of variance to the expected value of this number of storms are computed for each interval. These values are obtained both for the rainy days, with each rainy day considered as a storm, and for the storms, defined as uninterrupted sequences of rainy days, and for each of the three series of daily precipitation. Similarly, they are computed both for rainy hours, with each rainy hour considered as a storm, and for the storms, defined as uninterrupted sequences of rainy hours, for one series of hourly precipitation. The expected values (means) and variances of these numbers of storms are then divided by 13 days for daily rainfall series, or by 312 hours (13 x 24) for hourly rainfall series, in order to reduce these values to a unit of time. In this new form, they are considered as densities in time over the year. These densities of means are defined as λ_1-parameter. The 28 intervals and their corresponding densities of means, plotted for each interval, give a sufficient number of points for the study of variation of λ_1-parameter within the year. Similarly, the 28 densities of variance and the 28 values of ratio of the variance to the mean (or the 28 values of the ratio of the density of variance to λ_1) are plotted against time to show their variations within the year.

4.2 Daily precipitation series at Durango, Colorado. Figure 4.1 shows the properties of λ_1-parameter of rainy days as a function of time t, with each rainy day considered as a storm. Figure 4.2 gives the same properties of λ_1-parameter of storms which are defined as uninterrupted sequences of rainy days. Figure 4.3 shows the ratios of λ_1-parameter for the first definition (Fig. 4.1) and the second definition (Fig. 4.2) of storms.

Upper graphs of Figs. 4.1 and 4.2, lines (1), show λ_1 to follow periodic movements. Mainly, a 365-day basic harmonic is present as significant in both figures, given as lines (2), while Fig. 4.1 also shows the fifth harmonic (73 days) to be significant. It is evident that the 95% probablity level in Fisher's test of significance of harmonics does not show a very good fit.

In both cases, central graphs of Figs. 4.1 and 4.2 and their lines (3) show that the densities of variance have no significant harmonics. Instead of periodic movements, the average value of these densities is given as lines (4) in these figures.

Lines (5) in Figs. 4.1 and 4.2 give the ratios of densities of variance and λ_1 values, while lines (6) show the fitted periodic components of significant harmonics. It should be expected that a ratio of two parameters, one with no significant periodic movement and the other with periodic movement of small amplitudes, would have either small amplitudes, when significant harmonics are shown, or would show no significant harmonics. This is the case with lines (5) and (6) in Figs. 4.1 and 4.2, with relatively small amplitudes of significant harmonics.

For a Poisson distribution to be applicable to the distribution of the number of storms in an interval, it should be expected for the density of variance to be equal to λ_1. Their ratios should be close to unity with no significant harmonics. The results presented in lower graphs of Figs. 4.1 and 4.2 do not confirm either of these two expected properties of ratios. The average ratio is 1.672 for rainy days, and 0.641 for storms, as shown by lines (5) and (6) in Figs. 4.1 and 4.2, respectively. The ratios of fitted periodic component of Fig. 4.1, line (6), fluctuate between 1.4 and 2.0, for rainy days, and between 0.58 and 0.70 for storms as shown by periodic components of Fig. 4.2, line (6).

Deviations of the above ratios from unity, which in the case of rainy days are above unity and in the case of storms are below the unity, may be the result of definitions of storms used in this study for the types of data available. In the first case of rainy days being considered as storms, there are more rainy days than there are true number of storms. In the second case of storms being defined as uninterrupted sequences of rainy days, the true number of storms seems to be greater than the number of storms determined as uninterrupted sequences. In the first case, the densities of variance of Fig. 4.1, lines (3) and (4), may be larger than the densities of variance of the true number of storms. In the second case of Fig. 4.2, the results may be converse. These differences between the ratios of Figs. 4.1 and 4.2 are shown to be systematic, because they appear also in the other three stations, as shown later in the text.

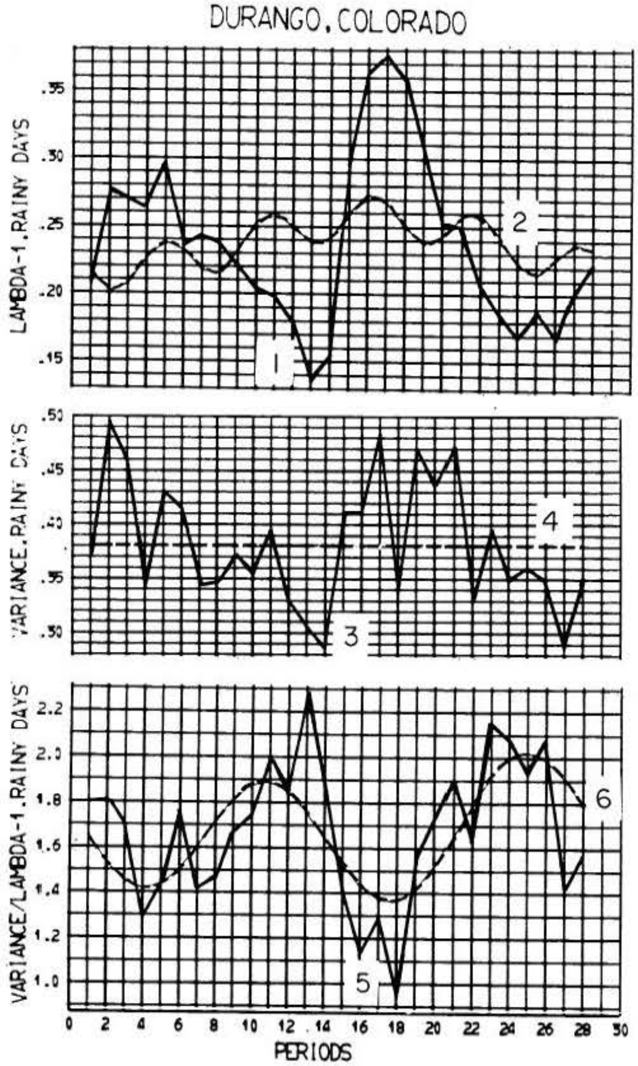

Fig. 4.1 Properties of λ_1-parameter (density of number of storms in time) for daily precipitation series at Durango, Colorado, with each rainy day considered as a storm: (1) computed λ_1 values; (2) periodic component of significant harmonics fitted to λ_1 values; (3) computed density of variance of the number of storms per interval; (4) average density of variance; (5) ratios of the density of variance to λ_1, and (6) periodic component of significant harmonics fitted to the ratio of density of variance to λ_1.

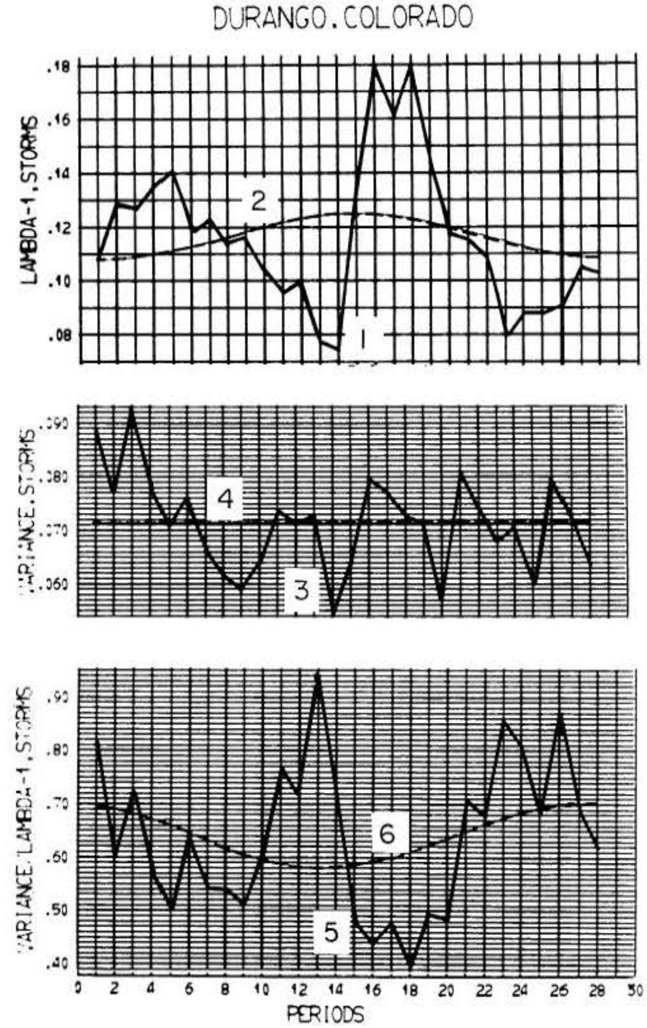

Fig. 4.2 Properties of λ_1-parameter for daily precipitation series at Durango, Colorado, with storms defined as uninterrupted sequences of rainy days: (1) computed λ_1 values; (2) periodic component of significant harmonic fitted to λ_1 values; (3) computed density of variance of the number of storms per interval; (4) average density of variance; (5) ratio of the density of variance to λ_1; and (6) periodic component of significant harmonic fitted to the ratio of density of variance to λ_1.

As it concerns the second departure from the expected patterns related to the ratio of densities of variance to the λ_1-parameter values, the amplitudes of fitted periodic components for Durango precipitation data are relatively small. The other stations do not show these periodicities in ratios at all, so the case of Durango daily precipitation series and the small periodicity may be assumed as being a product of sampling variation with the small probability of occurrence, rather than to be a systematic pattern.

Figure 4.3 shows that the ratios of the mean number of rainy days per interval to the mean number of storms per interval, as defined above, fluctuate in a narrow band. The mean ratio of 28 interval values is 2.034. The average duration of storms, defined as uninterrupted sequences of rainy days, is about two days.

Table 4.1 shows the main properties of the λ_1-parameter and of other parameters of the number of storms per interval for the Durango Station, computed from 28 interval values. Ratios of amplitudes of fitted periodic components to the means of parameters, as shown in Table 4.1, vary from 7.4% to 19.6%. For λ_1, they are 15.4% and 7.4%. Periodic components in λ_1 and in other parameters of the number of storms per interval for Durango are of relatively small practical significance. The parameter λ_1 is close to being a constant.

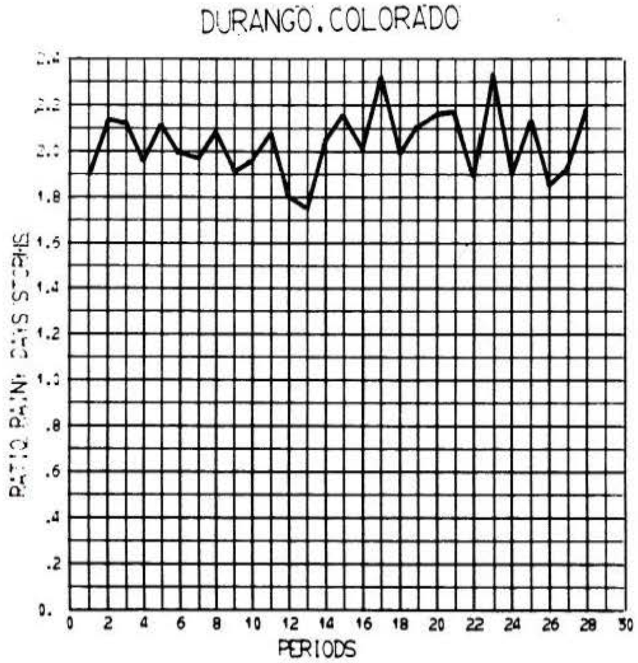

Fig. 4.3 Ratio of λ_1 of Fig. 4.1 (λ_1 for rainy days considered as storms) to λ_1 of Fig. 4.2 (λ_1 for storms defined as uninterrupted sequences of rainy days), for daily precipitation at Durango, Colorado. The average ratio is 2.034.

The question of the expected value of λ_1^*, where λ_1^* are true values of λ_1, if storms would be given as intermittent events and not as either the rainy days or uninterrupted sequences of rainy days, is discussed in Subchapter 4.6. The fact that the first approach of definition of storms gives $E(\lambda_1) = 0.238$, and the second approach yields $E(\lambda_1) = 0.116$ points out that $E(\lambda_1^*)$ should be somewhere between 0.116 and 0.238.

4.3 Daily precipitation series at Fort Collins, Colorado. Figure 4.4 shows the properties of λ_1-parameter of rainy days as a function of time t, with each rainy day considered a storm. Figure 4.5 gives the properties of λ_1-parameter of storms which are defined as uninterrupted sequences of rainy days. Figure 4.6 shows the ratios of λ_1-parameter for the first (Fig. 4.4) and the second definition (Fig. 4.5) of storms.

Upper graphs of Figs. 4.4 and 4.5, lines (1), show λ_1 to clearly follow periodic movements. Only the 365-day basic harmonic is shown as significant in both figures as lines (2). The oscillations of lines (1) around lines (2) may be considered as the pure sampling variation. Fisher's tests of significant harmonics show good fits.

The central graph of Fig. 4.4, line (3), shows also a periodic movement for the density of variance, while line (4) represents the fit of the periodic component with only the basic 365-day harmonic being significant. However, the central graph of Fig. 4.5, line (3), does not show any significant harmonic in the density of variance for the second definition of storms.

TABLE 4.1

PROPERTIES OF λ_1-PARAMETER AT DURANGO

Parameter	Definition of Storms	Expected Value	Variance	Amplitude of Periodic Component	Ratio of Amplitude to Expected Value
λ_1	Rainy days	0.238	0.0038	0.0367	0.154
	Storms	0.116	0.0008	0.0086	0.074
Density of Variance	Rainy days	0.382	0.0034	-	-
	Storms	0.071	0.0001	-	-
Ratio of Density of Variance to λ_1	Rainy days	1.672	0.0965	0.3291	0.196
	Storms	0.641	0.0203	0.0612	0.096

Lower graphs of Figs. 4.4 and 4.5, lines (5), give the ratios of densities of variance to λ_1 values. Lines (6) give the averages of these ratios. There is no significant harmonics in these ratios. It meets one of the basic conditions for the number of storms in an interval to follow the Poisson distribution. The average ratios are: 1.496 and 0.669, respectively for the two definitions of storms. These are the same patterns as for the previous series of daily precipitation. The same explanations for the average ratios not being unities in Figs. 4.4 and 4.5, lines (6), may be advanced as it was done for Figs. 4.1 and 4.2 .

Figure 4.6 shows that the ratios of the mean number of rainy days per interval to the mean number of storms per interval, as defined for Figs. 4.4 and 4.5, fluctuate in a relatively narrow range (1.4 - 2.1), though a periodicity is not excluded. The average ratio of 28 interval values is 1.76.

Table 4.2 shows the main properties of λ_1-parameter and the other parameters of the number of storms per interval for the Fort Collins Station, computed from 28 interval values. Ratios of amplitudes of fitted periodic components to the means of λ_1-parameter

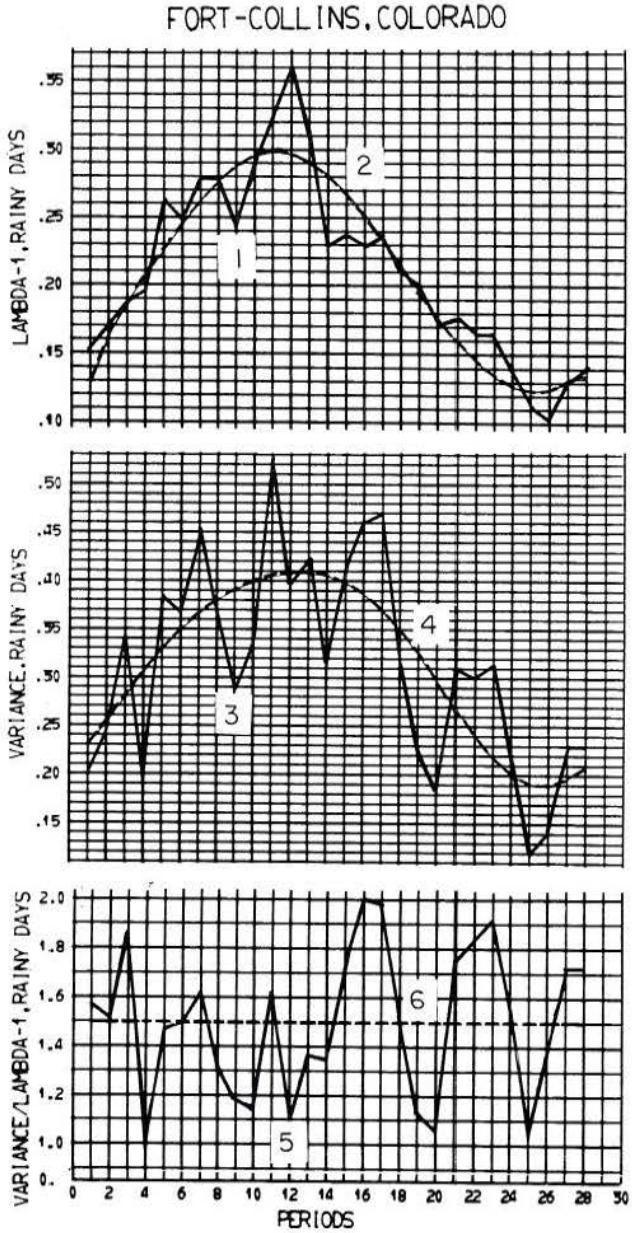

Fig. 4.4 Properties of λ_1-parameter (the density of the number of storms in time) for daily precipitation series at Fort Collins, Colorado, with each rainy day considered as a storm: (1) computed λ_1 values; (2) periodic component of significant harmonics fitted to computed λ_1 values; (3) computed densities of variance to the number of storms per interval; (4) periodic component of significant harmonics fitted to the densities of variance; (5) ratios of the density of variance to λ_1; and (6) average ratio of density of variance to λ_1.

in the two cases of rainy days and storms are 41.3% and 29.7%, respectively. Periodic components in λ_1 are relatively large, so that λ_1 for the Fort Collins Station is highly periodic. A similar ratio of 35.0% was shown for the density of variance for rainy days. It was zero for the density of variance of storms because it has no significant harmonics.

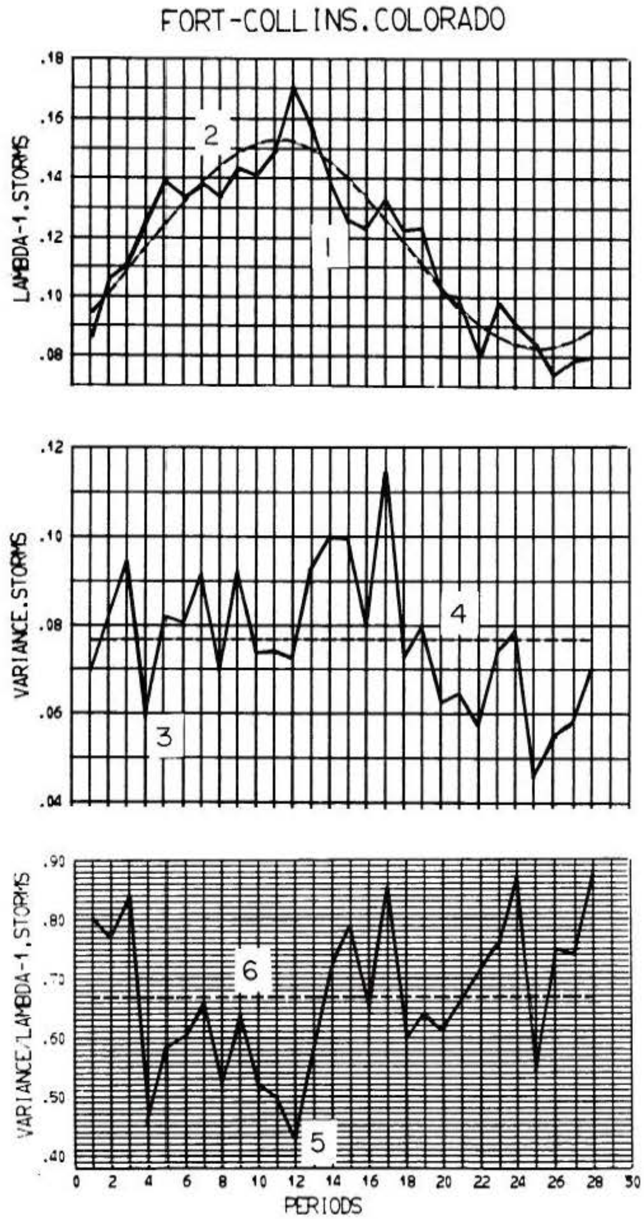

Fig. 4.5 Properties of λ_1-parameter for daily precipitation series at Fort Collins, Colorado, with storms defined as uninterrupted sequences of rainy days: (1) computed λ_1 values; (2) periodic component of significant harmonics fitted to computed λ_1 values; (3) computed densities of variance of the number of storms per interval; (4) the average of computed densities under (3); (5) ratios of the density of variance to λ_1; and (6) average ratio of density of variance to λ_1.

The question of the expected value of λ_1^*, for λ_1^* being the true λ_1 values, in case the storms are given as intermittent events of continuous rainfall intensity during any storm, is discussed in Subchapter 4.6. As $E(\lambda_1) = 0.211$ for rainy days and $E(\lambda_1) = 0.118$ for storms, in the first and second definition of storms, the expected value, $E(\lambda_1^*)$, of the true number of storms should be somewhere between 0.118 and 0.211.

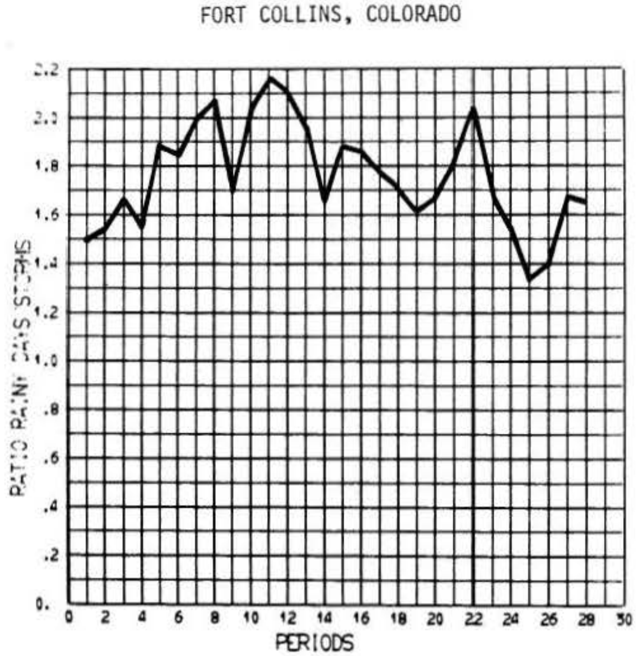

Fig. 4.6 Ratio of λ_1 of Fig. 4.4 (λ_1 for rainy days considered as storms) and λ_1 of Fig. 4.5 (λ_1 for storms defined as uninterrupted sequences of rainy days), for daily precipitation at Fort Collins, Colorado. The average ratio is 1.760.

4.4 Daily precipitation series at Austin, Texas. Figure 4.7 shows the properties of λ_1-parameter of rainy days, as a function of time t, with each rainy day considered as a storm. Figure 4.8 gives the same properties of λ_1-parameter of storms which are defined as uninterrupted sequences of rainy days. Figure 4.9 shows the ratios of λ_1-parameter for the first (Fig. 4.7) and the second definition (Fig. 4.8) of storms.

Upper graphs of Figs. 4.7 and 4.8, lines (1), show λ_1 to follow periodic movements. Lines (2) represent the fitted periodic components of significant harmonics. The sampling variations in lines (1) do not permit a detection of higher harmonics with a sufficient reliability. Therefore, a small difference in significant values of g, by Fisher's tests, makes one higher harmonic included in line (2) of Fig. 4.7 while another higher harmonic in line (2) of Fig. 4.8.

Central graphs of Figs. 4.7 and 4.8, lines (3), show no significant harmonics in densities of variance. The average values are presented in Figs. 4.7 and 4.8 as lines (4). The second central moments which underly these densities, have a much larger sampling variation than the first moments about the origin for lines (1) of Figs. 4.7 and 4.8. Therefore, these differences in sampling variations may explain why the tests for the densities of variance may not show the significant harmonics of periodic component while they show for λ_1-parameter.

Similarly for the lower graphs, the ratios of the density of variance to λ_1, as shown in lines (5) of Figs. 4.7 and 4.8, do not have any significant harmonics.

TABLE 4.2

PROPERTIES OF λ_1-PARAMETER AT FORT COLLINS

Parameter	Definition of Storms	Expected Value	Variance	Amplitude of Periodic Component	Ratio of Amplitude to Expected Value
λ_1	Rainy days	0.211	0.0044	0.0874	0.413
	Storms	0.118	0.0007	0.0350	0.297
Density of Variance	Rainy days	0.312	0.0107	0.1094	0.350
	Storms	0.077	0.0002	-	-
Ratio Density of Variance to λ_1	Rainy days	1.496	0.0874	-	-
	Storms	0.669	0.0159	-	-

Lines (6) give the average values of these ratios as 1.485 for rainy days considered as storms, and 0.672 for storms defined as uninterrupted sequences of rainy days. The conditions of the ratios of density of variance to λ_1 not showing any periodicity is fulfilled, assuming that the number of storms in an interval follows the Poisson distribution. However, the average ratios of 1.485 and 0.672 depart from the expected values of unity, as it was discussed in the two previous examples.

Figure 4.9 shows that the ratios of the mean number of rainy days per interval to the mean of storms per interval, or ratios of λ_1 of Fig. 4.7, line (1), to λ_1 of Fig. 4.8, line (1), fluctuate also in a narrow band of 1.43 - 2.08. The average ratio of 28 interval values is 1.764. The average duration of storms, defined as uninterrupted sequences of rainy days, is about 1 and 3/4 days.

Table 4.3 shows the main properties of λ_1-parameter and the other parameters of the number of storms per interval for the Austin Station, computed from 28 interval values. Ratios of amplitudes of fitted periodic components to the means of λ_1-parameter in the two cases of rainy days and storms are 20.4% and 24.1%, respectively. The periodic components in λ_1 are

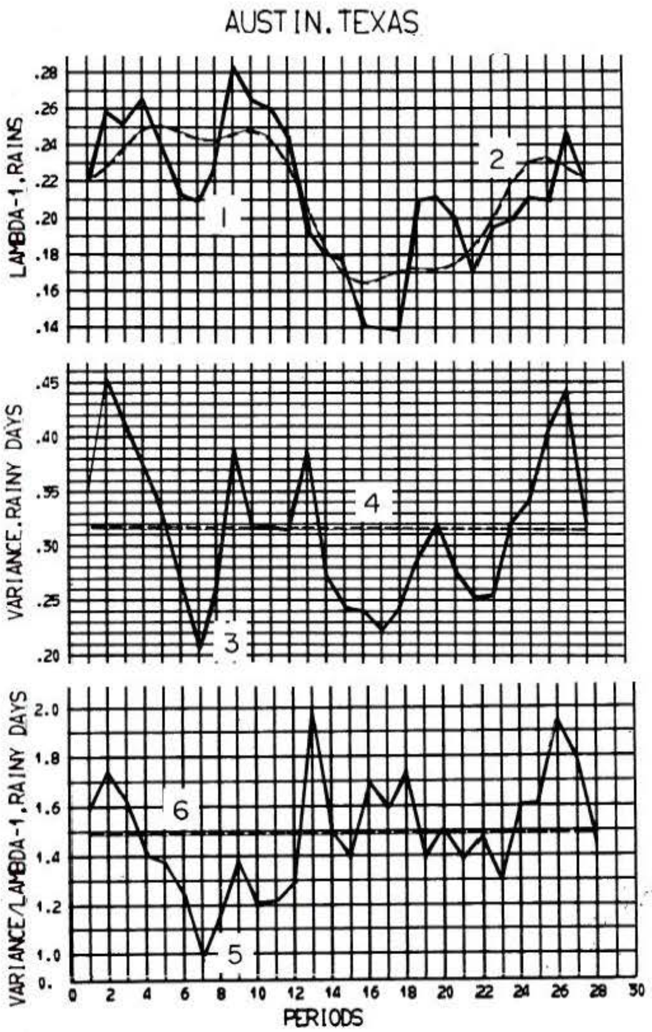

Fig. 4.7 Properties of λ_1-parameter (density of number of storms in time) for daily precipitation series at Austin, Texas, with each rainy day considered as a storm: (1) computed λ_1 values; (2) periodic component of significant harmonics fitted to λ_1 values; (3) computed densities of variance of the number of storms per interval; (4) average density of variance; (5) ratios of the density of variance to λ_1; and (6) average ratio of the density of variance to λ_1.

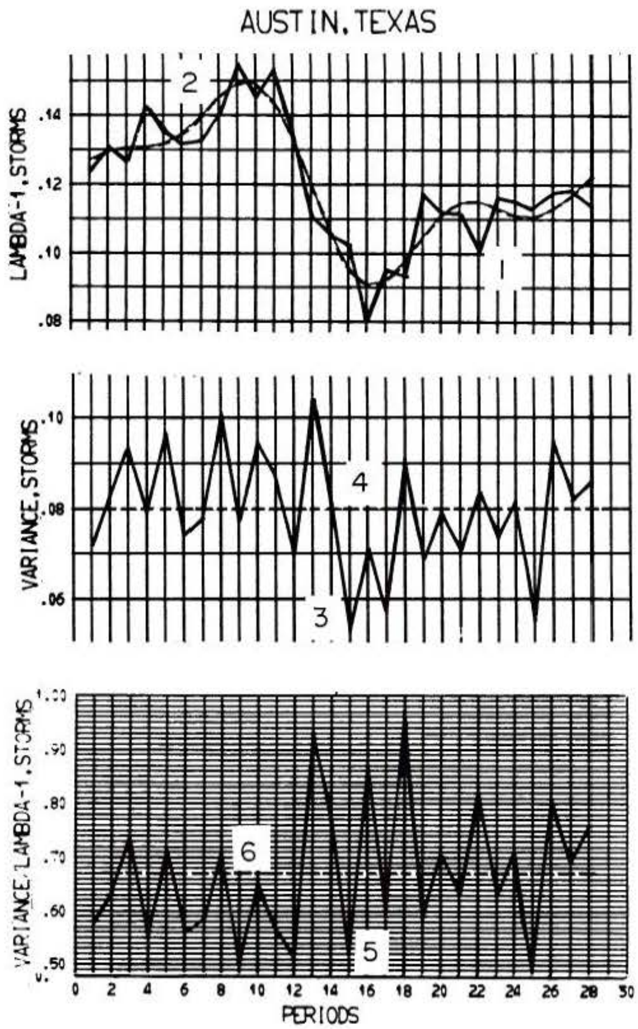

Fig. 4.8 Properties of λ_1-parameter for daily precipitation series at Austin, Texas, with storms defined as uninterrupted sequences of rainy days: (1) computed λ_1 values; (2) periodic component of significant harmonics fitted to λ_1 values; (3) computed densities of variance of the number of storms per interval; (4) average density of variance; (5) ratios of the density of variance to λ_1 and (6) average ratio of the density of variance to λ_1.

sufficiently large, so that λ_1 for the Austin Station can be assumed to be highly periodic. No periodic components are found in other parameters. The expected value of the true values λ_1^* for the Austin Station are discussed in Subchapter 4.6, as for other stations.

4.5 Hourly precipitation series at Ames, Iowa. Figure 4.10 shows the properties of λ_1-parameter for rainy hours, as a function of the time t, with each rainy hour considered as a storm. Figure 4.11 gives the same properties of λ_1-parameter of storms which are defined as uninterrupted sequences of rainy hours. Figure 4.12 shows the ratios of λ_1-parameter for the first (Fig. 4.10) and the second definitions (Fig. 4.11) of storms.

Upper graphs of Figs. 4.10 and 4.11, lines (1), show λ_1 to clearly follow periodic movements. Only the annual basic harmonic is present as significant in both figures, as shown by fitted periodic components of significant harmonics of lines (2). Neither the densities of variance, lines (3), nor the ratios of these densities to λ_1-parameter, lines (5), demonstrate any significant harmonic in Figs. 4.10 and 4.11. Lines (4) and (6) give the average values. The patterns and their explanations are similar to those of the previous example for Austin.

The average ratios of the density of variance to λ_1, of 8.892 for rainy hours, with each rainy hour considered as a storm, and of 2.350 for storms, and defined

TABLE 4.3

PROPERTIES OF λ_1-PARAMETER AT AUSTIN

Parameter	Definition of Storms	Expected Value	Variance	Amplitude of Periodic Component	Ratio of Amplitude to Expected Value
λ_1	Rainy days	0.214	0.0015	0.0436	0.204
	Storms	0.121	0.0003	0.0292	0.241
Density of Variance	Rainy days	0.315	0.0045	-	-
	Storms	0.080	0.0002	-	-
Ratio Density of Variance to λ_1	Rainy days	1.485	0.0551	-	-
	Storms	0.672	0.0166	-	-

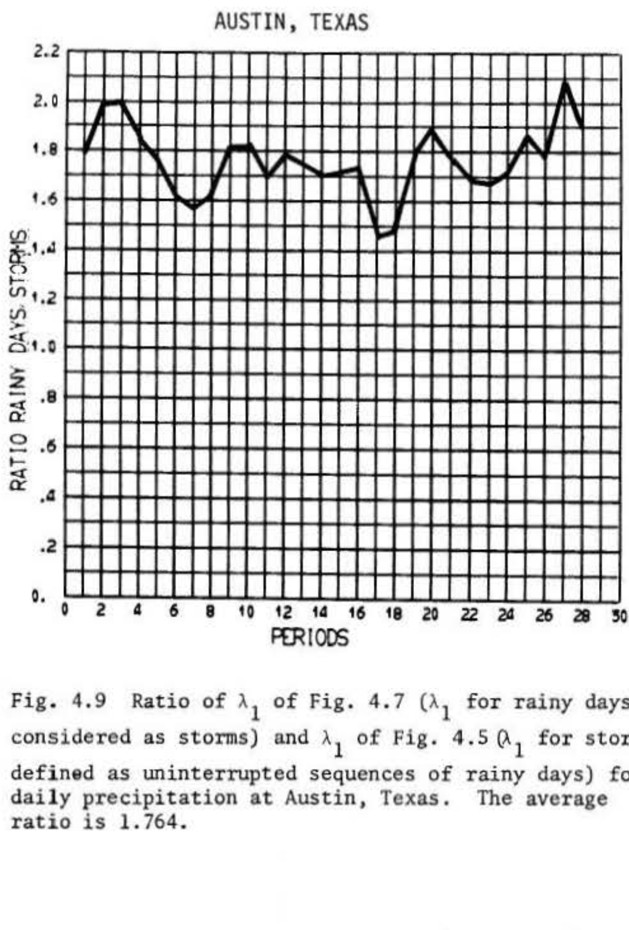

Fig. 4.9 Ratio of λ_1 of Fig. 4.7 (λ_1 for rainy days considered as storms) and λ_1 of Fig. 4.5 (λ_1 for storms defined as uninterrupted sequences of rainy days) for daily precipitation at Austin, Texas. The average ratio is 1.764.

as uninterrupted sequences of rainy hours, merit a special discussion which is given in the next subchapter. It relates also to the previous three examples of Durango, Fort Collins, and Austin daily precipitation series.

Figure 4.12 shows that the ratios of the mean number of rainy hours per interval to the mean number of storms per interval, or ratios of λ_1 of Fig. 4.10, line (1), to λ_1 of Fig. 4.11, line (1), also fluctuate in a narrow band of 2.25 - 3.60, except for a value at the 25th interval (5.22). The average ratio of 28 interval values is 3.11. Therefore, the average duration of storms, defined as uninterrupted sequences of rainy hours, is about three hours.

Table 4.4 shows the main properties of λ_1-parameter and the other parameters of the number of storms per interval, for hourly precipitation at the Ames Station, computed from 28 interval values. Ratios of amplitudes of fitted periodic components to the means of λ_1-parameter for both the rainy hours and storms as shown in Fig. 4.10 and 4.11, vary from 29.7% to 32.1%. This represents a clear periodicity in λ_1-parameter at the Ames Station. As the expected values of λ_1 are $E(\lambda_1) = 0.0535$ and $E(\lambda_1) = 0.0175$ for the two cases, the expected value of λ_1^*, obtained from the true numbers of storms in an interval, should be between these two expected values. If multiplied by 24, they refer to λ_1 per day instead per hour. This item is discussed in detail in the next subchapter.

4.6 Discussion of ratios of variance to the mean of number of storms in each interval. Figures 4.1, 4.2, 4.4, 4.5, 4.7, 4.8, 4.10 and 4.11, lines (5), give the ratios of the variance to the mean of the number of storms for each interval, with this number as the random variable. These ratios are identical to ratios of the density of variance (as given by lines (3) of these figures) to the λ_1 values (as given by lines (1) of the same figures) and for both definition of storms.

The basic derivation in this paper is that the number of storms in a sufficiently small interval of time within the year should follow the Poisson distribution. In that case, the variance and the mean of the number of storms in each interval should be equal, or the ratios of the density of variance and λ_1 should be unities regardless of the position of the small interval. However, because of sampling variations, it should be expected for these ratios to fluctuate randonly about the value of unity. Out of the above eight

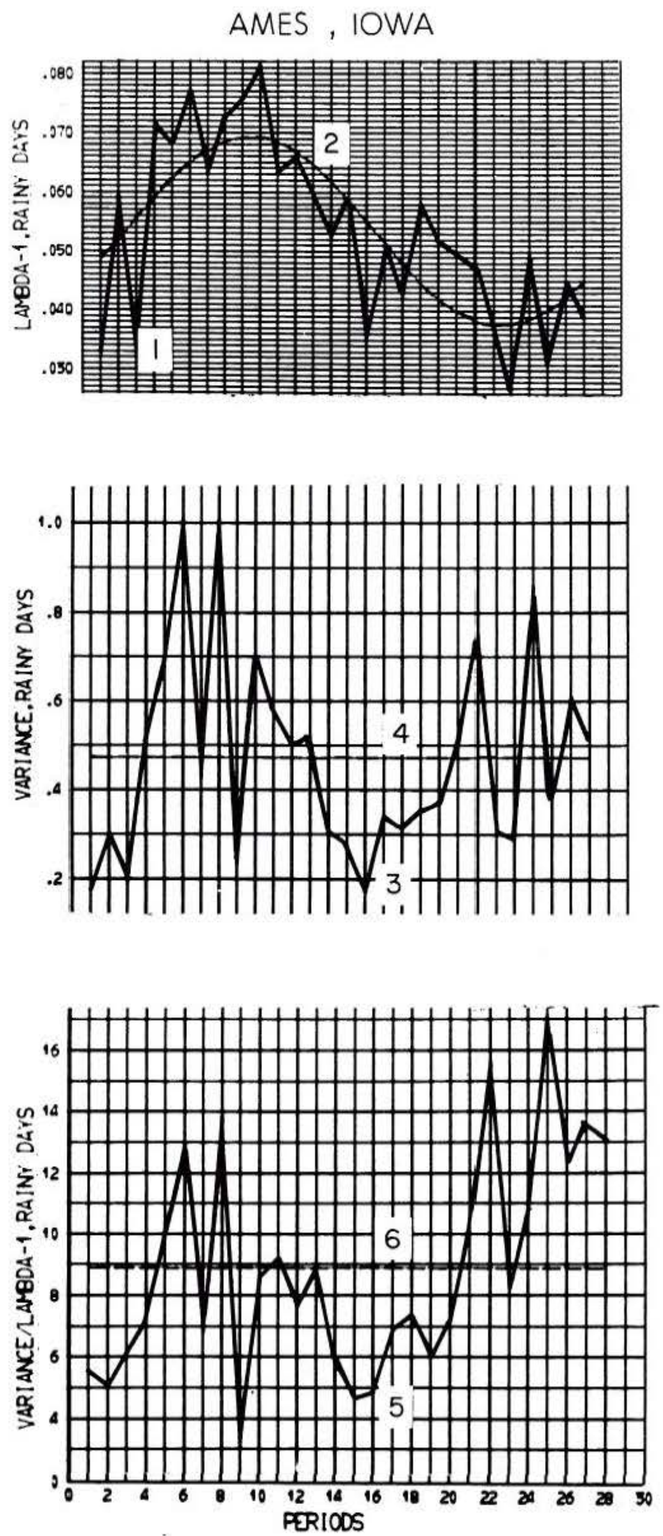

Fig. 4.10 Properties of λ_1-parameter (density of number of storms in time) for hourly precipitation series at Ames, Iowa, with each rainy hour considered as a storm: (1) computed λ_1 values; (2) periodic component of significant harmonic fitted to λ_1 values; (3) computed density of variance of the number of storms per interval; (4) average density of variance; (5) ratios of the density of variance to λ_1; and (6) average ratio of the density of variance to λ_1.

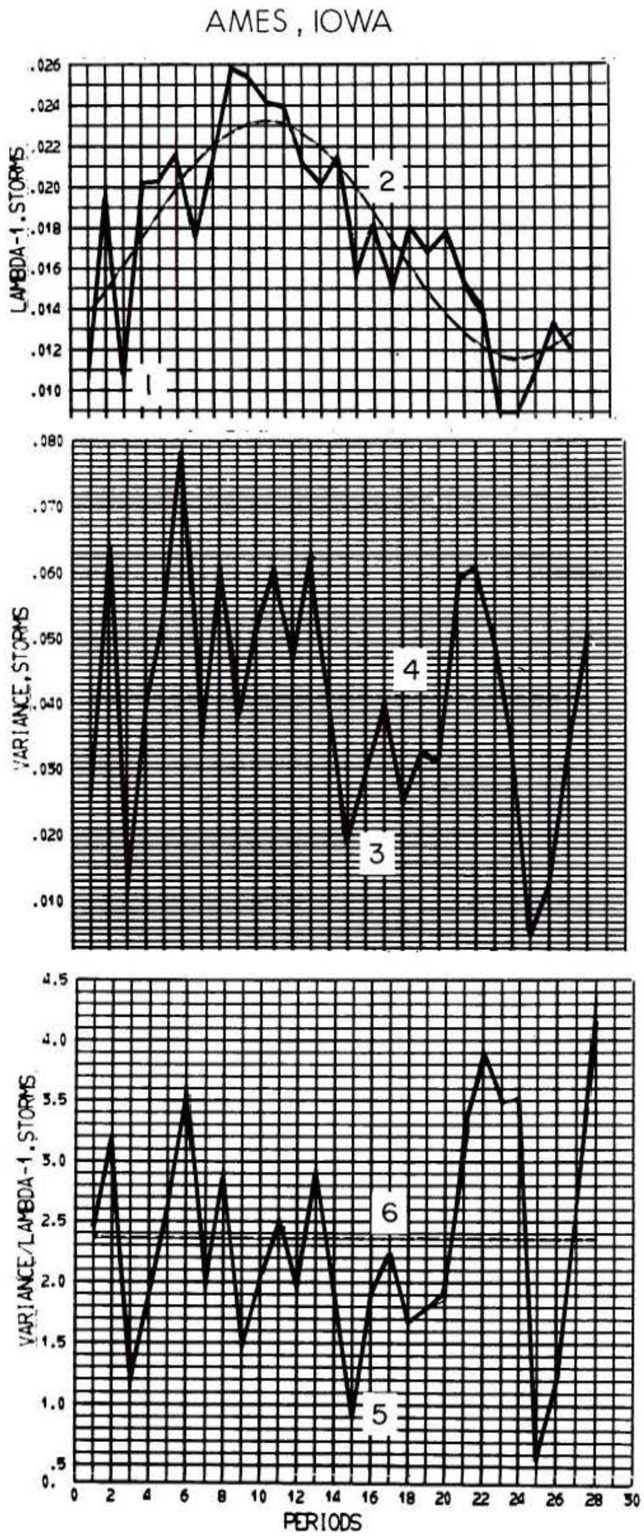

Fig. 4.11 Properties of λ_1-parameter for hourly precipitation series at Ames, Iowa, with storms defined as uninterrupted sequences of rainy hours: (1) computed λ_1 values; (2) periodic component of significant harmonic fitted to λ_1 values; (3) computed density of variance of the number of storms per interval; (4) average density of variance; (5) ratios of the density of variance to λ_1; and (6) average ratio of the density of variance to λ_1.

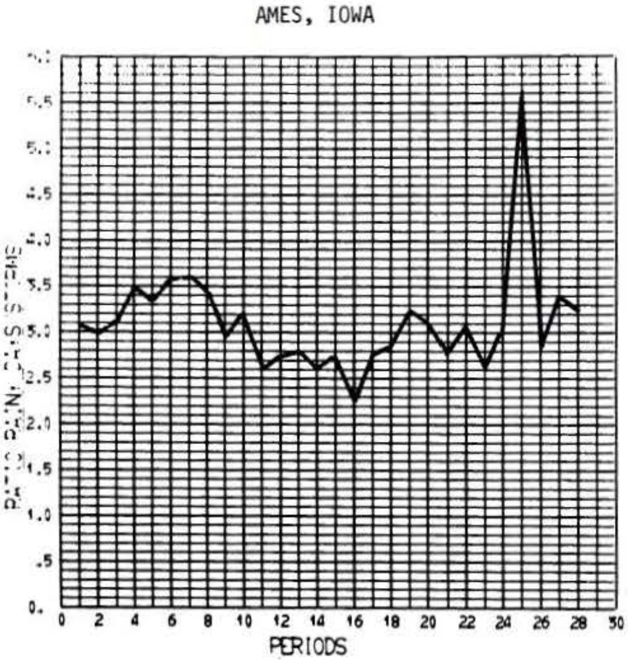

Fig. 4.12 Ratio of λ_1 of Fig. 4.10 (λ_1 for rainy hours considered as storms) and λ_1 of Fig. 4.11 (λ_1 for storms defined as uninterrupted sequences of rainy hours) for hourly precipitation at Ames, Iowa. The average ratio is 3.11.

figures for the four examples discussed, only the example of Durango daily precipitation series (Figs. 4.1 and 4.2) show a significant periodic component in these ratios. However, as it was discussed earlier, the amplitudes of these periodic components are relatively small. The other three examples and six figures (4.4, 4.5, 4.7, 4.8, 4.10, and 4.11) show these ratios not to be significantly different from constants. These constants as the average ratios (for Figs. 4.1 and 4.2 also as the average ratios) are not unities as it is expected for Poisson distribution.

The explanation for this discrepancy of expected and computed average ratios of the variance to the mean should be in the definition of the number of storms. Because data is given in form of daily and hourly values of precipitation, the number of storms as determined by either of the two definitions of storms may be different from the true number of storms.

The true number of storms is denoted by η_k. The definition of each rainy day or each rainy hour being considered as storms produce the number of storms, η_i, in each interval and for each year. Similarly, η_j is the number of storms defined as uninterrupted sequences of rainy days or rainy hours in each interval and for each year. Let denote η_i/η_k as ε_i and η_j/η_k as ε_j. The λ_1-parameter and the density of variance are defined as

$$\lambda_1 = \frac{1}{n\cdot\Delta t}\sum_{i=1}^{n}\eta_i \tag{4.1}$$

and

$$v_d = \frac{1}{n\cdot\Delta t}\sum_{i=1}^{n}(\eta_i-\lambda_1\Delta t)^2 , \tag{4.2}$$

where n = number of years and Δt = length of an interval. By replacing η_i by $\varepsilon_i\eta_k$, then

$$\lambda_1 = \frac{\varepsilon_i}{n\cdot\Delta t}\sum_{k=1}^{n}\eta_k \tag{4.3}$$

and

$$v_d = \frac{\varepsilon_i^{\,2}}{n\cdot\Delta t}\sum_{i=1}^{n}\left(\eta_k - \frac{1}{n\Delta t}\sum_{k=1}^{n}\eta_k\right)^2 . \tag{4.4}$$

The ratio

$$\frac{v_d}{\lambda_1} = \frac{\sum_{i=1}^{n}(\eta_i-\lambda_1\Delta t)^2}{\sum_{i=1}^{n}\eta_i} = \varepsilon_i \cdot \frac{\sum_{k=1}^{n}\left(\eta_k - \frac{1}{n\Delta t}\sum_{k=1}^{n}\eta_k\right)^2}{\sum_{k=1}^{n}\eta_k} \tag{4.5}$$

TABLE 4.4

PROPERTIES OF λ_1-PARAMETER AT AMES

Parameter	Definition of Storms	Expected Value	Variance	Amplitude of Periodic Component	Ratio of Amplitude to Expected Value
λ_1	Rainy hours	0.0535	0.0002	0.0159	0.297
	Storms	0.0175	0.0000	0.0058	0.321
Density of Variance	Rainy hours	0.4724	0.0537	-	-
	Storms	0.0410	0.0003	-	-
Ratio of Density of Variance to λ_1	Rainy hours	8.8920	12.6152	-	-
	Storms	2.3503	0.8800	-	-

or it is ε_i-times greater than the ratio in the case of true number of storms.

Similarly, in the second definition of storms, ε_j represents the constant to be determined as was ε_i in eq. (4.5). Therefore, if ε_i and ε_j are computed, they can be used as average values to reduce the number of storms, η_i or η_j, of the two definitions of storms, or to obtain the true number of storms. Table 4.5 summarizes values of ε_i and ε_j for the four examples in this study.

Designating the true value of λ_1 by λ_1^* and of v_d by v_d^*, then

$$\frac{v_d^*}{\lambda_1^*} = 1 = \left(\frac{v_d}{\lambda_1}\right) \cdot \frac{1}{\varepsilon_i} , \qquad (4.6)$$

for the first definition, and ε_j in eq. 4.6 for the second definition of storms. The average values, $E(\lambda_1)$, as given in Tables 4.1 through 4.4 for each of the two cases, are divided by ε_i or ε_j, whichever is relevant, and the average values of λ_1^*, $E(\lambda_1^*)$, are given in Table 4.5 computed by

$$E(\lambda_1^*) = \frac{E(\lambda_1)}{\varepsilon_i} , \qquad (4.7)$$

with ε_i replaced by ε_j in the second definition of storms. In the case of hourly data at Ames, the average values of $E(\lambda_1^*)$ are multiplied by 24 and are given in the last row of Table 4.5, in order to compare them with $E(\lambda_1^*)$ values for the other three stations. The surprising result is that $E(\lambda_1^*)$ in the first case is about 0.143, while in the second case it is about

TABLE 4.5

REDUCTION COEFFICIENTS FOR COMPUTATION

Station	Precipitation Data	ε_i	ε_j	$E(\lambda_1^*) = \frac{E(\lambda_1)}{\varepsilon_i}$	$E(\lambda_1^*) = \frac{E(\lambda_1)}{\varepsilon_j}$
Durango	daily	1.672	0.641	0.142	0.181
Fort Collins	daily	1.485	0.672	0.142	0.176
Austin	daily	1.485	0.672	0.144	0.180
Ames	hourly	8.892	2.350	(0.0060)	(0.0075)
Ames	for λ_1 multiplied by 24 hours			0.144	0.181

0.180, which are very consistent numbers in each case. An average of the two is somewhere around $E(\lambda_1^*) = 0.160$. However, more research should be done on many station examples before these consistent patterns in the two sets of $E(\lambda_1^*)$ values can be explained. To meaningfully explain these consistent patterns in $E(\lambda_1^*)$, obtained for the two cases of definition of storms, precipitation data of continuous rainfall intensities during the intermittent storms, concurrently with hourly and daily precipitation data, would be required from several precipitation stations.

To obtain the true numbers of storms from daily or hourly precipitation data, the numbers of storms per interval should be divided either by ε_i or ε_j, whichever is relevant. In other words, the bias introduced in the numbers of storms, by using the conventional data of hourly and daily precipitation, may be corrected by dividing these numbers by a constant, ε. This procedure is based on the conclusion of Chapter II that the true number of storms is Poisson-distributed, with the mean and variance equal.

To test whether the number of storms in an interval is a Poisson-distributed random variable, using the hourly or daily data, the procedure should be, one or the other, or both of the above two definitions of storms should be used. Then the λ_1-paramter should be computed, and ε_i or ε_j, or their mean should be determined as the average ratio between the apparent number and the true number of storms. Then all numbers of storms, computed for each interval and for each of n years, should be divided by ε. The distributions of the new number of storms should then be tested, whether they are well fitted by Poisson distributions. The finite values of λ_1^* to be used are the reduced values of $\lambda_1^* = \lambda_1/\varepsilon$, where λ_1 are computed values from daily or hourly rainfall values, and ε is the relevant reduction coefficient.

CHAPTER V

YIELD CHARACTERISTIC OF STORMS

5.1 Significance and computation of the parameter of yield characteristic of storms. The analysis of six random variables, which are discussed in Chapter II as functions of the stochastic process $\{\xi_t\}$, has shown that some of their distributions are dependent on λ_2. The parameter λ_2 is the yield characteristic of storms given by eq. (2.16) in Chapter II. The procedure for computing it is in the following text. The basic conclusion in this analysis is that the existence of periodicity in the $\{\xi_t\}$ -process imposes a periodicity in the parameter λ_2, with the year as the basic period, of λ_2 depends on time. In other words, the annual periodicities in the mean, in the standard deviation and in some other parameters of $\{\xi_t\}$ -process, for various intervals of the year, are reflected also as the periodicity of λ_2 -parameter. This is important because there is often a conviction among hydrometeorologists that the average water yield per storm is very close to being independent of seasons (or of date position in the year). Through λ_2 all other random variables, which are functions of $\{\xi_t\}$ -process and dependent on λ_2, should exhibit a similar periodicity as the parameter λ_2.

Properties of λ_2 -parameter are studied in this chapter for the four examples of precipitation data as described in Chapter III, Durango, Fort Collins, Austin and Ames. Computations of this parameter are made in four different ways, and in turn for each of the two definitions of storms, except for the rainy hours, with each considered as a storm, of Ames Station. For each interval, the total precipitation of the first rainy day, after the interval begins, is determined for each of n years. This gives, for that interval, a distribution of the precipitation amount of the first rain, x_1, with the expected value, $E(x_1)$. Similarly, the total precipitation amounts of the first two rainy days, the first three rainy days, and the first fifteen rainy days, after the interval begins, are determined for each of n years, as well as their corresponding expected values, $E(x_2)$, $E(x_3)$ and $E(x_{15})$. For storms defined as uninterrupted sequences of rainy days or rainy hours, the total precipitation amounts, X_i, of the first storm, the first two storms, the first three storms, and the first ten storms, after the interval begins, are determined for each of n years, with $i = 1, 2, 3$ and 10, as well as their expected values, $E(x_1)$, $E(x_2)$, $E(x_3)$ and $E(x_{10})$. As intervals are only 13 days long, it is clear that the 15th rainy day or the 10th storm of a given interval, as defined above, always fall outside the corresponding interval, while the third or the second rainy day or storm often may occur either inside or outside of an interval. Even the first rainy day or the first storm falls sometimes to the right of an interval, whenever a 13-day interval has no rain.

This will be evident from the material presented in Chapter VII.

If the number of storms taken into the computation of the total precipitation amount x_ν is $\nu = 1, 2,$ 3 and 15 for rainy days, and $\nu = 1,2,3,$ and 10 for storms defined as uninterrupted sequences of rainy days or rainy hours, and if the expected value (or the mean) over n years of these total precipitation amounts are $E(x_\nu)$, then λ_2 as the parameter describing the yield characteristic of storms in time is defined by

$$\lambda_2 = \left(\frac{\nu}{E(x_\nu)}\right) . \tag{5.1}$$

The yield characteristic of storms is inversely proportional to the average yield per storm, for a given time of the year. Instead of λ_2, the use of $E(x_\nu)/\nu$ may also be feasible. However, λ_2 suits better various integrals in the distributions of functions of $\{\xi_t\}$ -process, like eqs. (2.41, 2.51 and 2.52). One must be aware that the greater λ_2 the smaller is the average yield of storms. Values of λ_2 are attached to a position of the year, in this case to the beginning of each interval, so that it can be studied as a function of time. By the use of eq. (5.1) and four different ν values, four graphs of 28 values of λ_2 are obtained for 28 positions within the year. As the four examples of precipitation data show, the four series of λ_2 for different ν values are surprisingly similar in their general patterns.

5.2 Daily precipitation series at Durango, Colorado. Figure 5.1 shows four graphs of λ_2 as functions of the time t, which are composed of 28 values each representing the beginnings of intervals of 13 days, for $\nu = 1, 2, 3$ and 15, in the case each rainy day is considered as a storm. Figure 5.2 shows the similar graphs for $\nu = 1, 2, 3,$ and 10 in the case the storms are defined as uninterrupted sequences of rainy days.

All graphs of Figs. 5.1 and 5.2 are fitted by the periodic components of significant harmonics through the use of Fisher's tests. These eight graphs show that the sampling fluctuations of the computed λ_2 values about the fitted periodic components decrease with an increase of ν. For $\nu = 15$ in Fig. 5.1 and $\nu = 10$ in Fig. 5.2, the fitted periodic components of two significant harmonics (12-month and 6-month) have small differences from the computed λ_2 values. However, for $\nu = 1$ in Fig. 5.1, only the 12-month harmonic is significant, with large sampling fluctuation of computed λ_2 values about it, while for $\nu = 1$ in Fig. 5.2 no significant periodic component is shown, though general patterns are similar to those of the graph for $\nu = 2$ in Fig. 5.2 or to those of $\nu = 1$ in Fig. 5.1. The larger the sampling fluctuation about a periodic component, the less likely a harmonic becomes significant if the ratio of its amplitude to the

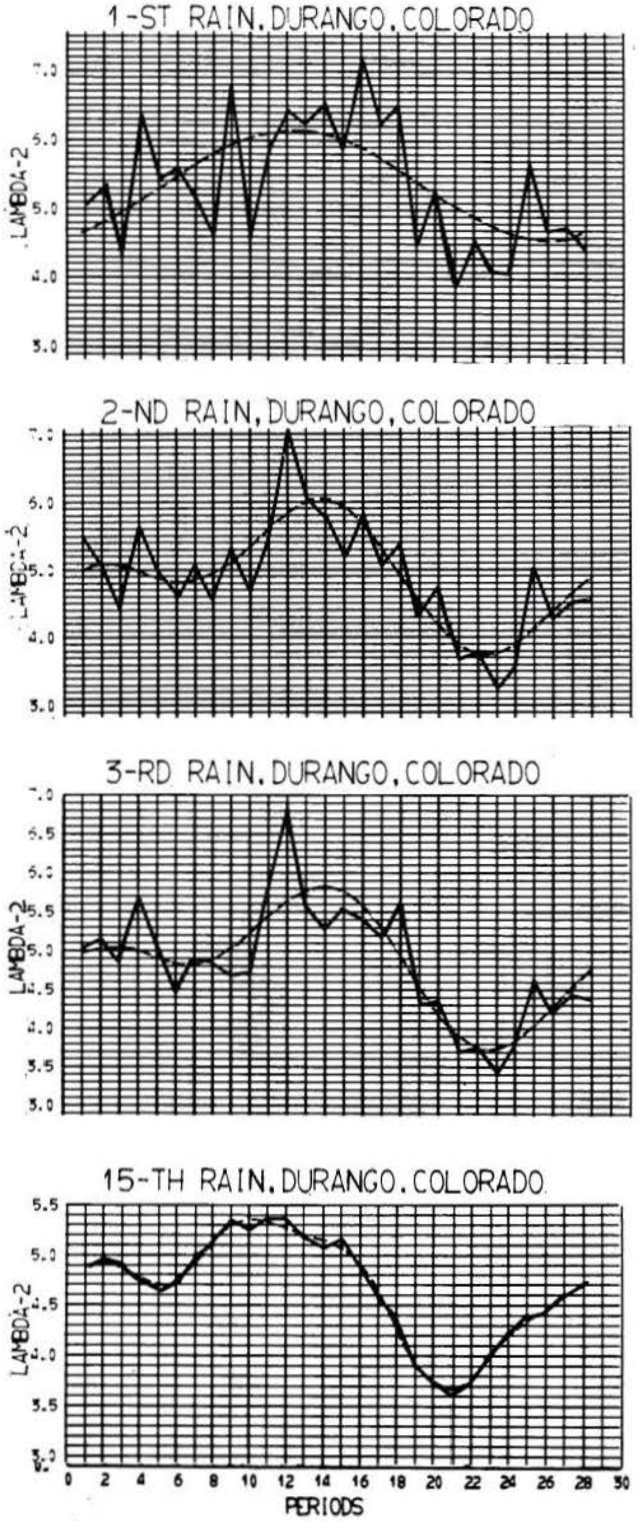

Fig. 5.1. The time function of λ_2 - parameter (yield characteristic of storms) for Durango, Colorado, with each rainy day considered as a storm. The four graphs relate to the number of rainy days, which are used in computing λ_2 (up to 1st rain, up to 2-nd rain, up to 3-rd rain and up to 15-th rain), and related to an interval.

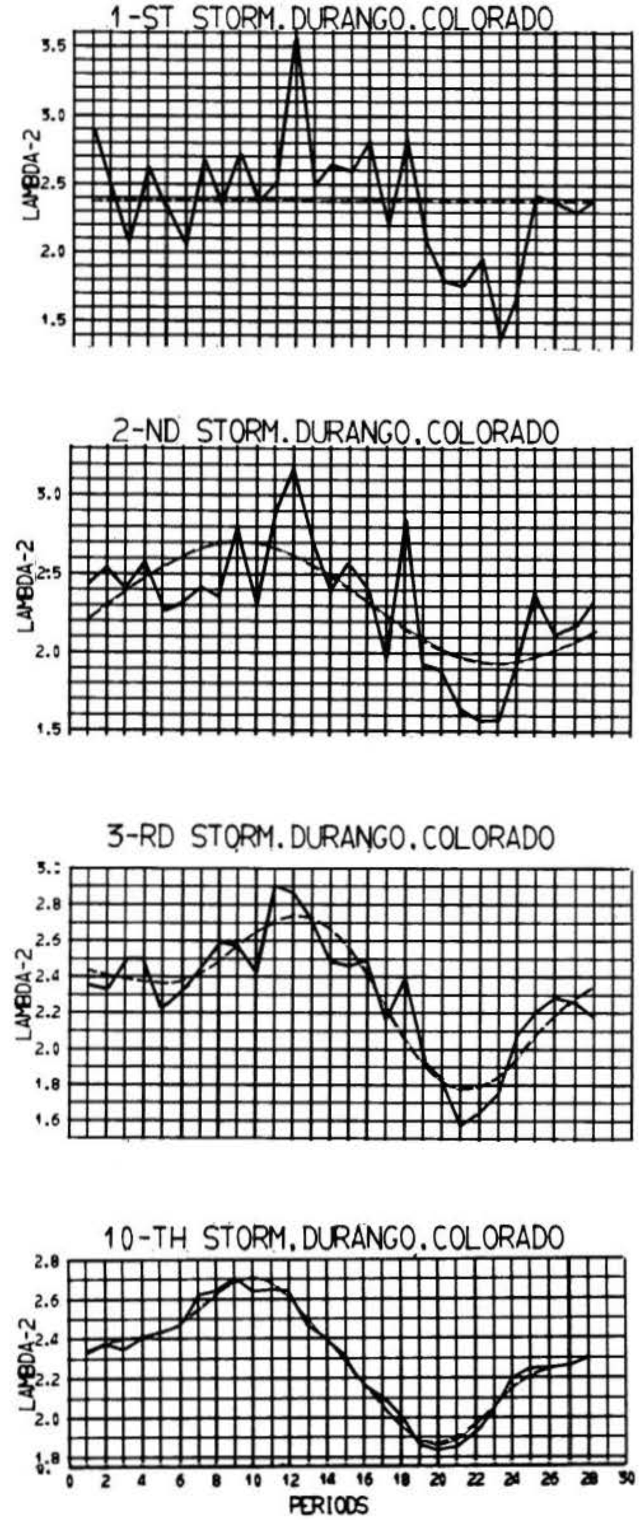

Fig. 5.2. The time function of λ_2 - parameter (yield characteristic of storms) for Durango, Colorado, with each storm defined as uninterrupted sequences of rainy days. The four graphs relate to the number of storms, which are used in computing λ_2 (up to 1-st storm, up to 2-nd storm, up to 3-rd storm and up to 10-th storm), and related to an interval.

mean value is relatively small. Therefore, the larger a value ν the smoother is λ_2 graph.

Various values λ_2, computed for $\nu = 15$ or $\nu = 10$, are not representative of the initial position of a given interval, thus creating a bias, because 15 rainy days or 10 storms may be spread over a large portion of the year and often extend into the next year. An attractive approach would be to use λ_2 values of $\nu = 3 - 4$. They are a compromise between the large sampling fluctuations around a smooth curve of λ_2 for $\nu = 1 - 2$, and a bias resulting from ν being very large, say $\lambda_2 > 5$. The Durango data extend over 71 years. For $\nu = 1$, there are 71 values of x_1 in the computation of $E(x_1)$, while there are $15 \cdot 71$, or $10 \cdot 71$ storms in the computation of $E(x_{15})$ or $E(x_{10})$, respectively for $\nu = 15$ of Fig. 5.1 and $\nu = 10$ of Fig. 5.2. This compromise between the sampling error and the bias of storms, extended over a much longer period than the interval length, looks as an unavoidable dilemma in order to produce λ_2 -function of time t with the least sampling error and a minimum of bias.

Amplitudes of periodic components in Figs. 5.1 and 5.2 are relatively small in comparison with the average value of λ_2. The ratios of amplitude to $E(\lambda_2)$ are given in Table 5.1 as well as the other basic properties of λ_2.

Parameters of Table 5.1 show the following properties. Values of $E(\lambda_2)$, for $\nu = 1$, 2, 3 and 15 or $\nu = 1$, 2, 3 and 10, decrease with an increase of ν, from 5.35 to 4.674, in case of Fig. 5.1, and from 2.372 to 2.301 in case of Fig. 5.2. However, differences are small, because the average of four means of λ_2 are 4.942 and 2.324 respectively for Figs. 5.1 and 5.2. The decrease may be explained partly by the fact that $E(\lambda_2)$ value is the harmonic mean of $E(x_\nu)/\nu$.

The yield characteristic of storms depends on how well the storm length is determined, because the definition of storm duration greatly influences the total storm precipitation, the greater storm duration, the greater the total precipitation. The discussion in Chapter IV has shown that the use of ε-ratios changes the number of storms in an interval by changing their durations. The average number of days or hours of storms is changed by ε in order that they come close to the average true number of days or hours in storms.

By using the ratio ε,

$$E(\lambda_2^*) = \frac{E(\lambda_2)}{\varepsilon} , \qquad (5.2)$$

with ε either ε_i or ε_j, which depends on the definition of storms, the expected value $E(\lambda_2^*)$, as the true storm characteristic is obtained. As $\varepsilon_i = 1.672$ and $\varepsilon_j = 0.641$, respectively for the two cases, then $E(\lambda_2)/\varepsilon$ are 2.95 and 3.62 as estimates of the true value $E(\lambda_2^*)$. Differences between 2.95 and 3.62 may be explained in the same manner as differences in $E(\lambda_1^*)$ have been when the two $E(\lambda_1)$ have been divided by ε_i and ε_j for the two definitions of storms. In other words, the storms are of smaller durations for the first definition of storms, and of larger duration for the second definition of storms for daily precipitation series, than are the true durations of storms. The true value $E(\lambda_2^*)$ is likely to be around 3.25 - 3.30.

As $E(x_\nu) = \nu/\lambda_2$, $E(x_\nu)$ is then ν times the harmonic mean of λ_2. Therefore, $E(\lambda_2)$ and $E(x_\nu)$, $\nu = 1, 2, \ldots$, are not reciprocal values. This fact, combined with a smaller fluctuation of λ_2 around its periodic component for large values of ν, may partly explain why $E(\lambda_2)$ decrease with an increase of ν.

Variances of λ_2 computed from 28 values of λ_2 for each case of the four values of ν show a rapid decrease with an increase of ν. It complies with the expected decrease of sampling fluctuations of λ_2 about the periodic component with an increase of ν. However, this decrease of var λ_2, with an increase of ν, may also be partly due to the harmonic central second moment, as shown by eq. (5.3) in later text.

Ratios $E(\lambda_2)/\varepsilon$ show to be much closer among themselves for the eight values of $E(\lambda_2)$, in the two

TABLE 5.1

PROPERTIES OF λ_2-PARAMETER AT DURANGO

Figure	ν	$E(\lambda_2)$	Variance of λ_2	Amplitude C_λ	$C_\lambda/E(\lambda_2)$		$E(\lambda_2)/\varepsilon$
5.1	1	5.350	0.844	0.794	0.150	1.672	3.20
	2	4.903	0.654	1.139	0.233		2.93
	3	4.843	0.555	1.063	0.219		2.90
	15	4.674	0.262	0.866	0.184		2.79
5.2	1	2.372	0.189	0.000	0.000	0.641	3.70
	2	2.322	0.157	0.377	0.162		3.63
	3	2.302	0.105	0.488	0.212		3.60
	10	2.301	0.065	0.428	0.186		3.59

cases of Figs. 5.1 and 5.2, because a division by ε forces the two cases closer to the true duration of storms. Ratios of the amplitudes C_λ of periodic component of λ_2 and the expected value $E(\lambda_2)$ is about 15.0% - 23.3%. This relatively small amplitude-mean ratio may be the reason why often λ_2 or its inverse is assumed to be a constant in practical investigation of storm yields.

5.3 Daily precipitation series at Fort Collins, Colorado. Figure 5.3 shows four graphs of λ_2 as functions of the time t, which are composed of 28 values each representing an interval of 13 days, for ν = 1, 2, 3 and 15, in the case each rainy day is considered as a storm. Figure 5.3 shows similar graphs for ν = 1, 2, 3 and 10, in the case the storms are defined as uninterrupted sequences of rainy days.

All graphs of Figs. 5.3 and 5.4 have been fitted by the periodic components of significant harmonics. The same patterns in fluctuation of λ_2 exist for this example as for the previous example. The variation of λ_2 about the fitted periodic components decreases rapidly with an increase of ν. All eight graphs show two significant harmonics, both 12-month and 6-month, and in general the same patterns of λ_2. The change of λ_2 with time seems to be best represented in cases of $\nu = 2$ or $\nu = 3$.

Table 5.2 gives the basic properties of λ_2 -parameter at Fort Collins Station. The decrease of $E(\lambda_2)$ with ν in both cases of Figs. 5.3 and 5.4 may be partly explained by the fact that $E(\lambda_2)$ is the harmonic mean of $E(x_\nu)/\nu$. The variation of x_ν about its periodic component affects its harmonic mean. This may be a reason for the average value of $E(x_\nu)/\nu$ to be as good or even a better general characteristic of storm yields as $E(\lambda_2)$.

The variance of λ_2 decreases rapidly with an increase of ν, also in this example. It is largely a result of sampling variation decreasing with an increase of ν, about the periodic component. It may be because the variance of λ_2 is computed by the following expression

$$\text{var } \lambda_2 = \frac{1}{28} \sum_{k=1}^{28} \left[\frac{\nu}{E_k(x_\nu)} - \frac{1}{28} \sum_{k=1}^{28} \frac{\nu}{E_k(x_\nu)}\right]^2, \quad (5.3)$$

where $E_k(x_\nu)$ stands for the $E(x_\nu)$ value of the k-th 13-days interval. This equation involves the harmonic mean and the harmonic second central moment.

Ratios of the amplitude C_λ of periodic component of λ_2 to $E(\lambda_2)$ show a good consistency, and relatively large values in the eight graphs of Figs. 5.3 and 5.4. In the first case, they are 34.7% - 49.5%. In the second case, they are 46.8% - 73.4%. The periodic components in λ_2 for this example are the important property. The decrease of $C_\lambda/E(\lambda_2)$ with an increase of ν may be due to the definition of λ_2, as the inverse of the average storm yield.

The true expected values of λ_2^*, given as $E(\lambda_2^*) = E(\lambda_2)/\varepsilon$, vary in these two cases between 3.97 - 4.76, and 5.08 - 5.56, respectively. The likely average value of λ_2^* is around 4.70 - 4.90.

5.4 Daily precipitation series at Austin, Texas. Figure 5.5 shows four graphs of λ_2 as a function of time, with ν = 1, 2, 3 and 15, in the case each rainy day is considered as a storm. Figure 5.6 shows the similar graphs for ν = 1, 2, 3 and 10, in the case the storms are defined as uninterrupted sequences of rainy days.

All eight graphs of Figs. 5.5 and 5.6 are fitted by the periodic components. They also show that the sampling fluctuation of λ_2 about the fitted periodic components decreases with an increase of ν. All periodic components, except the first one in Fig. 5.6 for ν = 1, have two significant harmonics, both 12-month and 6-month, while the component of Fig. 5.6 for $\nu = 1$ has only a 12-month significant harmonic.

Table 5.3 gives the basic properties of λ_2 -parameter at Austin Station. The first four $E(\lambda_2)$ values seem to slightly decrease with an increase of ν, while the second four $E(\lambda_2)$ values seem to fluctuate randomly about its mean value. It is a significant result that $E(\lambda_2)$ of Fig. 5.5 and $E(\lambda_2)$ of

TABLE 5.2

PROPERTIES OF λ_2-PARAMETER AT FORT COLLINS

Figure	ν	$E(\lambda_2)$	Variance of λ_2	Amplitude C_λ	$C_\lambda/E(\lambda_2)$	ε	$E(\lambda_2)/\varepsilon$
5.3	1	7.125	6.209	3.526	0.495	1.496	4.76
	2	6.217	3.564	2.849	0.458		4.18
	3	6.185	3.626	2.805	0.453		4.14
	15	5.932	1.700	2.060	0.347		3.97
5.4	1	3.719	3.143	2.730	0.734	0.669	5.56
	2	3.654	2.456	2.379	0.651		5.46
	3	3.633	2.213	2.225	0.613		5.43
	10	3.396	0.928	1.588	0.468		5.08

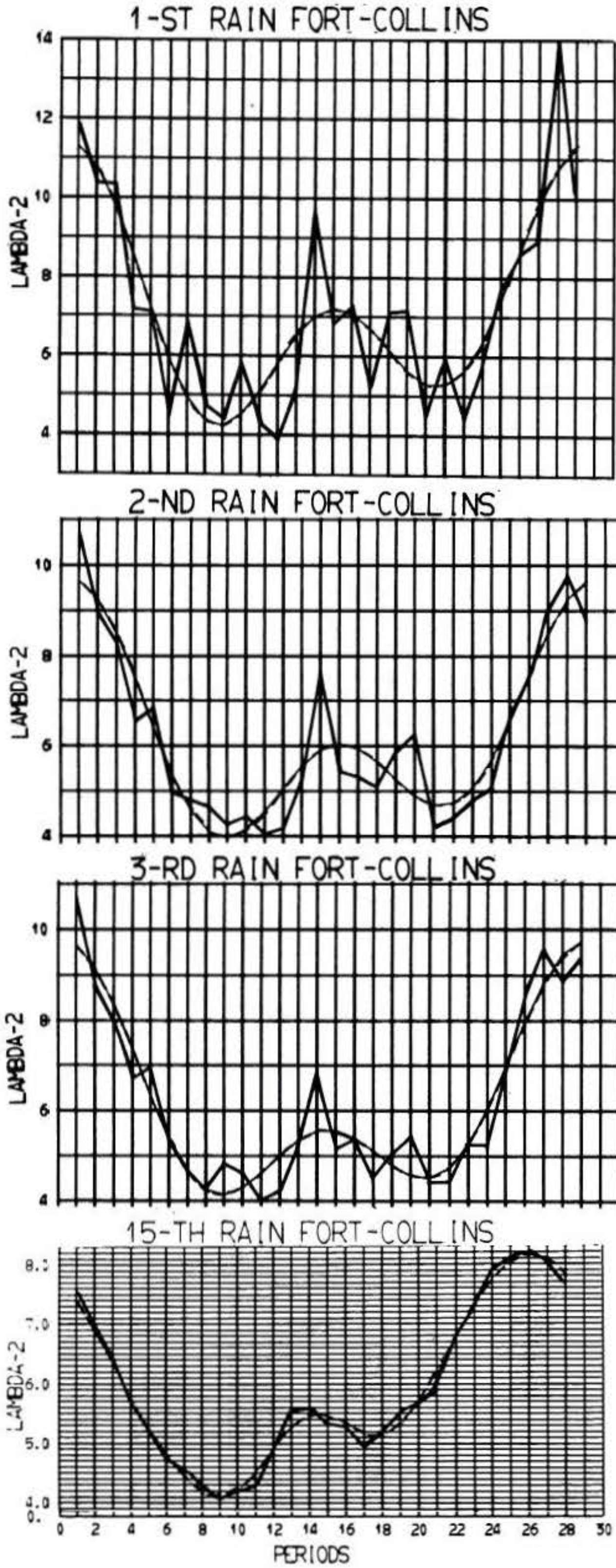

Fig. 5.3. The time function of λ_2 - parameter (yield characteristic of storms) for Fort Collins, Colorado, with each rainy day considered as a storm. The four graphs relate to the number of rainy days, which are used in computing λ_2 (up to 1st rain, up to 2-nd rain, up to 3-rd rain and up to 15-th rain), and related to an interval.

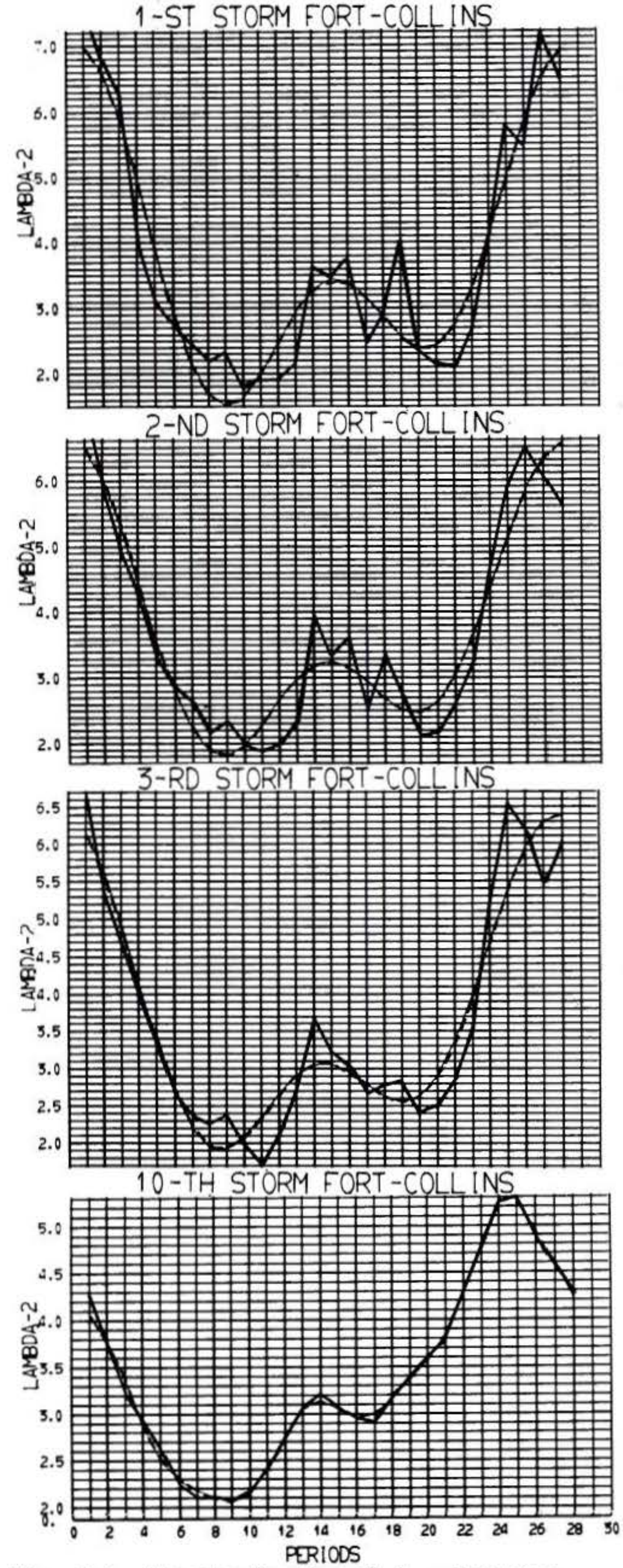

Fig. 5.4. The time function of λ_2 - parameter (yield characteristic of storms) for Fort Collins, Colorado, with each storm defined as uninterrupted sequences of rainy days. The four graphs relate to the number of storms, which are used in computing λ_2 (up to 1-st storm, up to 2-nd storm, up to 3-rd storm and up to 10-th storm), and related to an interval.

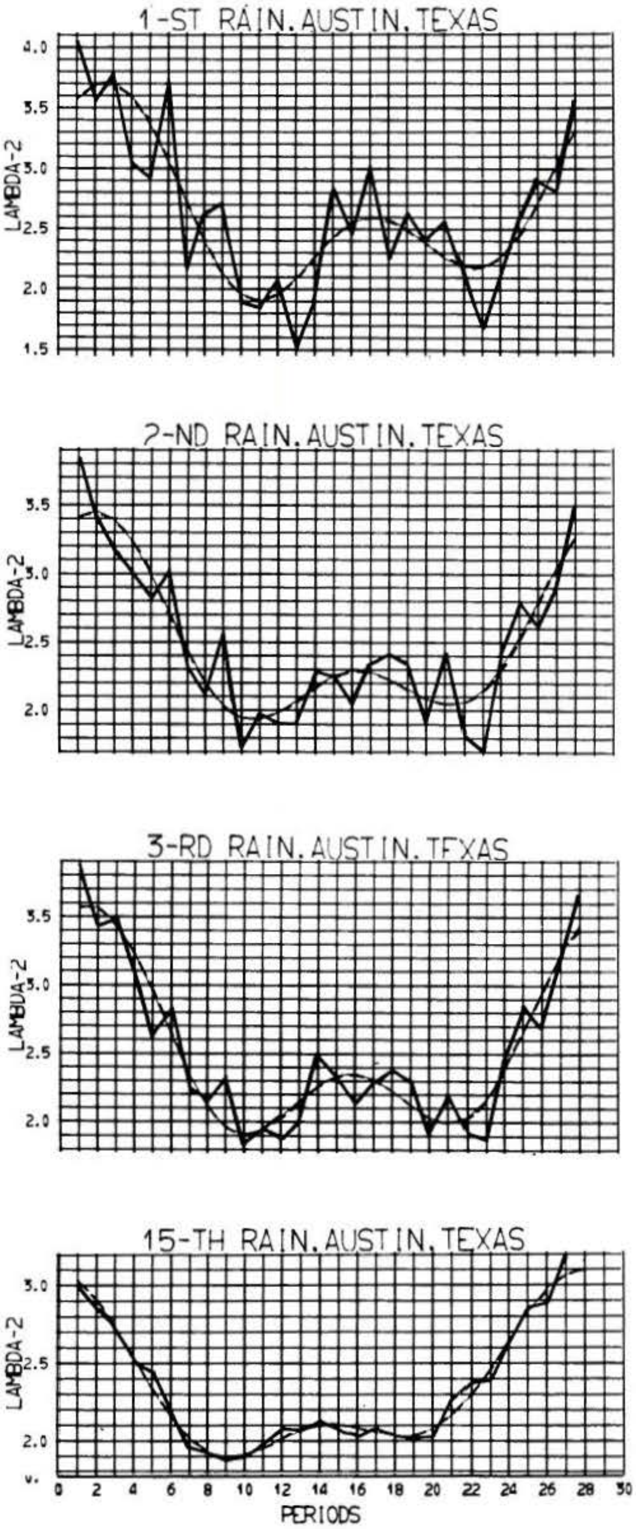

Fig. 5.5. The time function of λ_2 -parameter (yield characteristic of storms) for Austin, Texas, with each rainy day considered as a storm. The four graphs relate to the number of rainy days, which are used in computing λ_2 (up to 1st rain, up to 2-nd rain, up to 3-rd rain and up to 15-th rain), and related to an interval.

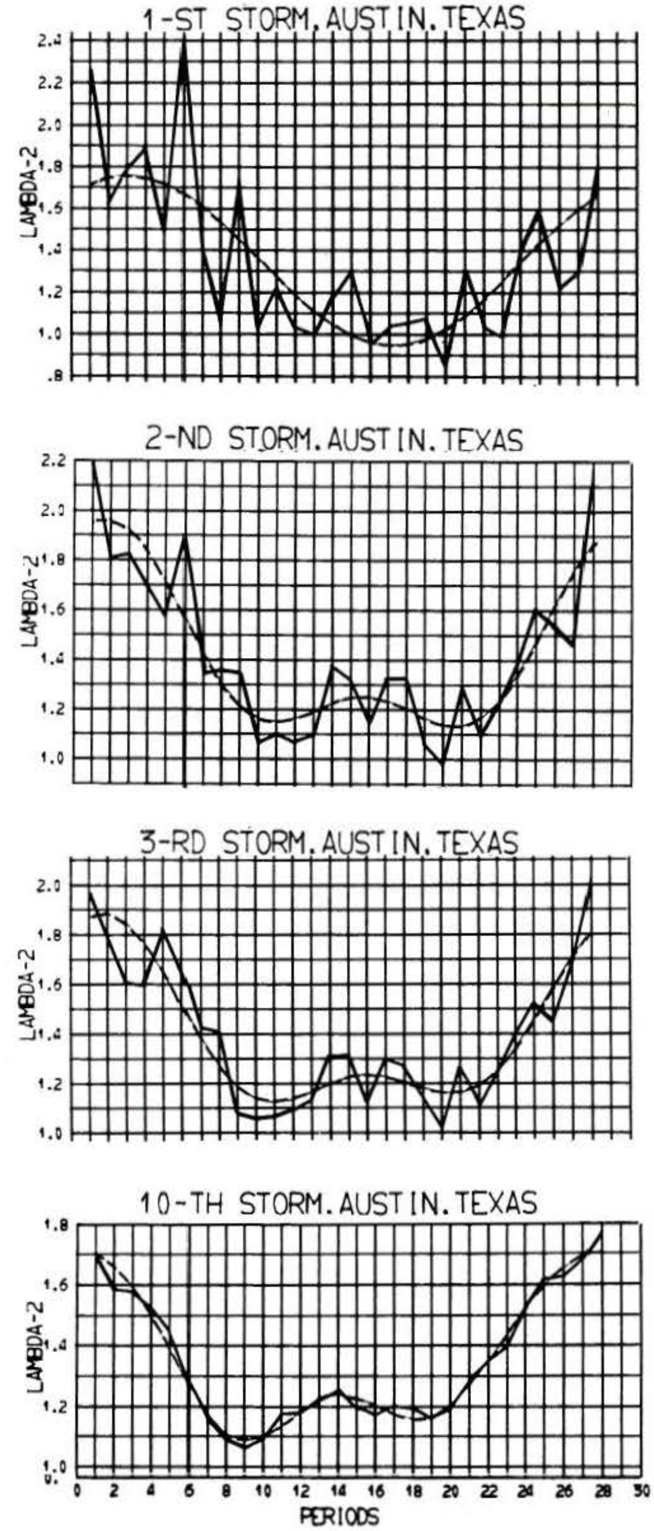

Fig. 5.6. The time function of λ_2 -parameter (yield characteristic of storms) for Austin, Texas, with each storm defined as uninterrupted sequences of rainy days. The four graphs relate to the number of storms, which are used in computing λ_2 (up to 1-st storm, up to 2-nd storm, up to 3-rd storm and up to 10-th storm), and related to an interval.

TABLE 5.3

PROPERTIES OF λ_2-PARAMETER AT AUSTIN

Figure	ν	$E(\lambda_2)$	Variance of λ_2	Amplitude C_λ	$C_\lambda/E(\lambda_2)$	ε	$E(\lambda_2)/\varepsilon$
5.5	1	2.639	0.428	0.906	0.343	1.485	1.78
	2	2.485	0.305	0.763	0.307		1.67
	3	2.515	0.329	0.831	0.331		1.69
	15	2.341	0.162	0.596	0.255		1.58
5.6	1	1.354	0.153	0.407	0.300	0.672	2.02
	2	1.417	0.106	0.416	0.294		2.11
	3	1.389	0.082	0.383	0.276		2.07
	10	1.348	0.045	0.314	0.233		2.01

Fig. 5.6 change little with the change of ν. This may be explained by somewhat smaller variations of λ_2 about its periodic components than in the previous cases.

The variance of λ_2 rapidly decreases with ν for Austin. The amplitudes of periodic components are relatively large, and their ratios to $E(\lambda_2)$ range from 23.3% to 34.3% or between 1/4 - 1/3, which is a significant variation. As λ_2 are much smaller between the 11th and 23rd 13-day interval than for the other intervals, the storms during these intervals have the largest water yields.

The use of ε-factor gives the true expected values, $E(\lambda_2^*)$, which range between 1.58 - 1.78 for the case of Fig. 5.5, and between 2.01 - 2.11 for the case of Fig. 5.6. The true value $E(\lambda_2^*)$ may be somewhere between 1.80-1.90. This is less than one half of the corresponding value for the Fort Collins Station. In other words, storms at Austin have more than twice the average storm yield than at Fort Collins.

5.5 Hourly precipitation series at Ames, Iowa. Figure 5.7 gives four graphs of λ_2 as a function of time, with ν = 1, 2, 3 and 10 for the case of storms defined as uninterrupted sequences of rainy hours. The study of each rainy hour being considered as a storm, like it was done for λ_1, is not carried out for λ_2.

All four graphs of Fig. 5.7 are fitted by the periodic components. For ν = 1 and ν = 3 only the 12-month harmonic is significant, while for ν = 2 and ν = 10 both the 12-month and 6-month harmonic is significant. The 6-month harmonic in Fisher's test for 95% probability level was, in these four cases, close to the critical value g_c, with two of them a little greater and two of them a little smaller.

Table 5.4 gives the basic properties of λ_2 -parameter at Ames Station, but only for storms as defined above. Both the expected values of λ_2, $E(\lambda_2)$, and the variance of λ_2 increase in this case with an increase of ν. These variations in $E(\lambda_2)$ and var λ_2 among the four graphs may be of random nature, because it is likely that most of these storms happen to occur within the 13-day interval. The amplitudes of fitted harmonics increase with an increase of ν. The ratio $C_\lambda/E(\lambda_2)$ is very high, 43.7% - 65.1%, which means the periodicity in λ_2 is its important property. The values of $E(\lambda_2^*)$, obtained by eq. (5.2), range between 2.64 - 2.94, which are relatively consistent values. This reversal of the trend of $E(\lambda_2)$ and var λ_2 with an increase of ν for hourly data may be eventually explained by two additional factors: (a) the sample size in this case is n = 18 years, while n = 71.69 and 70 in the previous three examples, and (b) the definition of storms in this case of hourly precipitation may bear on these results.

TABLE 5.4

PROPERTIES OF λ_2-PARAMETERS AT AMES

Figure	ν	$E(\lambda_2)$	Variance of λ_2	Amplitude C_λ	$C_\lambda/E(\lambda_2)$	ε	$E(\lambda_2)/\varepsilon$
5.7	1	6.241	8.453	2.725	0.437	2.350	2.66
	2	6.215	9.482	3.995	0.643		2.64
	3	6.548	9.240	3.403	0.520		2.79
	10	6.917	11.001	4.502	0.651		2.94

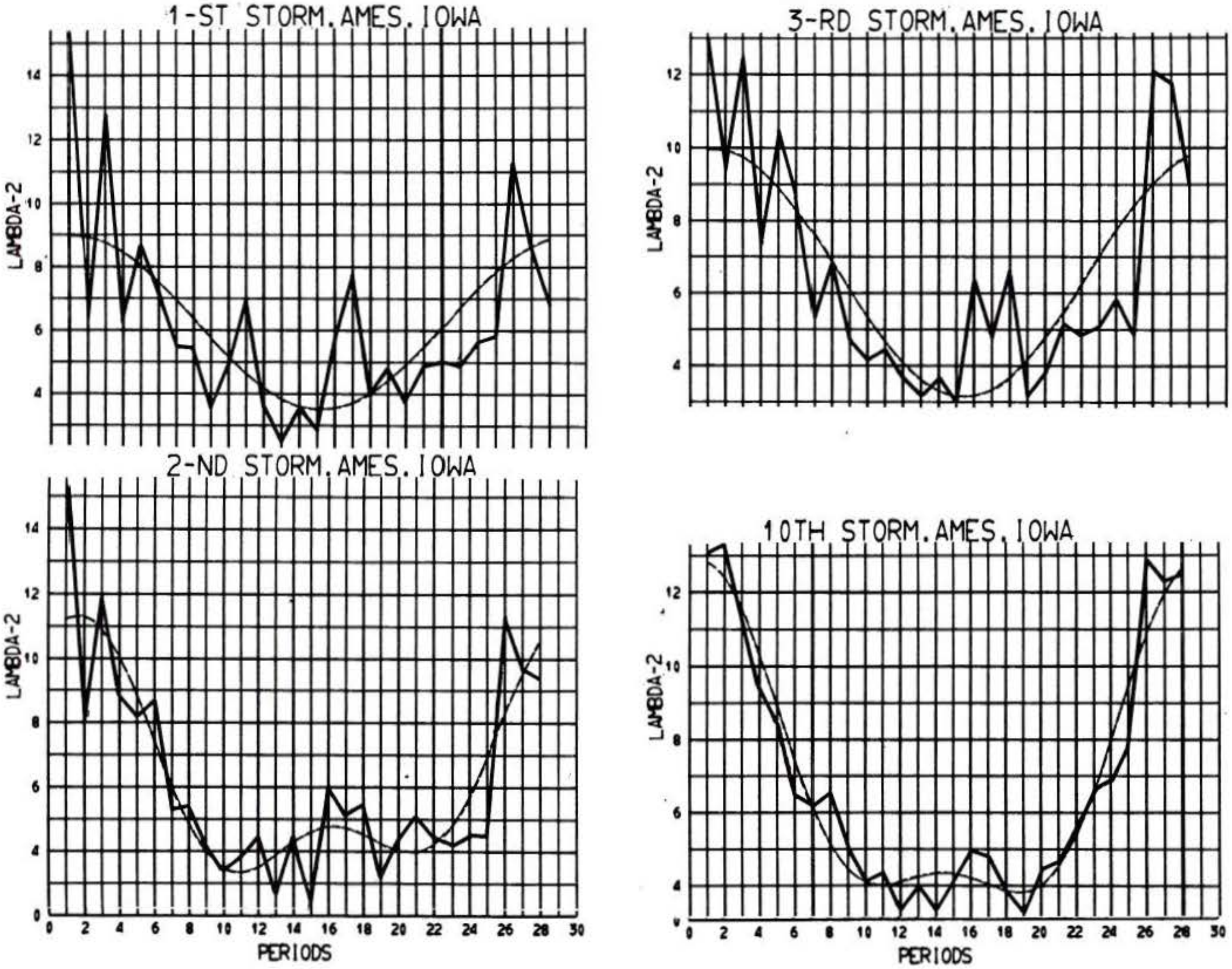

Fig. 5.7 The time function of λ_2-parameter (yield characteristic of storms) for Ames, Iowa, with each storm defined as uninterrupted sequences of rainy hours. The four graphs relate to the number of storms, which are used in computing λ_2 (up to 1-st storm, up to 2-nd storm, up to 3-rd storm and up to 10-th storm), and related to an interval.

5.6 Ratio λ_1/λ_2. The ratio of the two parameters, λ_1 and λ_2, represents the mean precipitation in the unit time interval, because

$$E(x_t) = \frac{\lambda_1 t}{\lambda_2} , \tag{5.4}$$

where t = the interval length, or

$$\frac{\lambda_1}{\lambda_2} = \frac{E(x_t)}{t} \tag{5.5}$$

The mean precipitation in the unit time interval is called the density of precipitation in time in the text that follows.

The comparison of Figs. 5.1 and 5.2 with Figs. 4.1 and 4.2, lines (2), demonstrates the significant harmonics of λ_2 and λ_1 periodic components at Durango Station not to be necessarily in phase. This may be because the amplitudes of periodic components in λ_1 and λ_2 are all relatively small in comparison with $E(\lambda_1)$ and $E(\lambda_2)$. Therefore, the parameters g of significant harmonics may be close to the critical value of g_c. The line (2) of the lefthand graph of Fig. 3.1 is equivalent to eq. (5.4), with t = 13 days, but it has the same shape as the ratio λ_1/λ_2 of eq. (5.5). It shows a small amplitude of a complex periodic component of the density of precipitation in time.

The comparison of Figs. 5.3 and 5.4 with Figs. 4.4 and 4.5, lines (2), for Fort Collins Station shows the significant harmonics of 12-month for λ_2 and λ_1 to be out of phase. This means the large λ_1 has a small counterpart in λ_2. As the average storm yields are the inverse of λ_2, this means that large densities of storms in time have at the same time, or approximately so, the large values of the average storm yield. The fact that λ_1 and λ_2 are out of phase at Fort Collins Station explains a very large amplitude in the line (2) of lefthand graph of Fig. 3.2 for the mean precipitation over 28 intervals.

When periodic components of λ_1 and λ_2 are out of phase, there is a large amplitude of periodicity in the density of precipitation, if λ_1 and λ_2 are sufficiently periodic. If the periodic components of λ_1 and λ_2 are in phase, it decreases the amplitude of perodic component in the mean precipitation. Therefore, it is possible to have λ_1 and λ_2 with significant periodic components, but the density of precipitation in time may not show a significant periodicity. The density of storms in time and the average yeild of storms in time may have compensating periodic components to produce non-periodic densities of precipitation.

The comparison of Figs. 5.5 and 5.6 with Figs. 4.7 and 4.8, lines (2), demonstrate a difference both in the shape and phases of periodic components of λ_1 and λ_2 parameters for Austin Station. The ratio of λ_1/λ_2, as equivalent within the multipling constant of 13, to the line (2) of the lefthand graph of Fig. 3.3, shows a relatively complex periodic movement for this station.

The comparison of Fig. 5.7 and Fig. 4.11, line (2), for the storms at Ames Station, with the storm defined as uninterrupted sequences of rainy hours, demonstrates a significant shift in phases of the λ_2 and λ_1 significant 12-month harmonics. The ratio λ_1/λ_2 is then very high, which is confirmed by the lefthand graph of Fig. 3.4, line (2).

Basically, the density of precipitation in time is decomposed in two factors, the density of storms in time, λ_1, and the yield of storms, λ_2. This approach provides more insight into the structure of stochastic-probability process of storm precipitation than the classical statistical parameters, as they change with the year.

5.7 Closing remarks. The above four examples of properties of λ_2 -parameter, as it change within the year, demonstrate clearly that the average storm yield depends on seasons. The eventual contention that $1/\lambda_2$ -values, as the average storm yields, are not significantly different from a constant, is not confirmed by the above four examples.

The method of computation of λ_2 by using the first ν storm for each interval with $\nu = 1, 2, 3, \ldots$, and for each ν by obtaining λ_2 as a function of time t, poses the problem of which λ_2 -graph should be used. For small ν, there is a great sampling variation of computed λ_2 about the periodic component. For large ν, the bias is unavoidable because many storms do not pertain to the position of the interval, though they are ν sequential storms after the interval begins. The suggestion in this study for using either $\nu = 3$ or $\nu = 4$ needs further study, and this problem needs to be investigated for a large number of stations. As an indirect and practical approach, the λ_2 -parameter should be computed by using eq. (5.4), or by

$$\lambda_2 = \frac{\lambda_1 t}{E(x_t)} \quad . \tag{5.6}$$

The main objection in this latter approach is the fact that λ_1 may have a large sampling fluctuation which is automatically transferred to λ_2. The sampling fluctuation of $E(x_t)$, about its true periodic component, should be relatively small, as $E(x_t)$ is the first moment of interval precipitation amount. The next objection may be the problem with the definition of storms, imposed by the availability of data in form of hourly and daily precipitation. This gives either shorter or longer duration, and therefore also either smaller or greater total precipitation of individual storms, than would be the case if data are given as the intermittent storms, each with the recorded rainfall intensity hyetograph.

Chapter VI

PROBABILITY DENSITIES OF STORM PRECIPITATION

6.1 Definition of storm precipitation. The function, X_ν, of the stochastic process $\{\xi_t\}$ is defined as the total precipitation for ν storms, in a sequence of storms. The probability density function of X_ν is derived in Chapter II and presented by eq. (2.52). The integral in that equation for $x_o = o$,

$$\Lambda_2(o,x) = \int_o^x \lambda_2(s)ds \tag{6.1}$$

represents the average number of storms necessary to produce the precipitation amount X. The question arises of how λ_2 should be determined; by the use of ν storms, which are close to the number η_x of storms which produce the precipitation amount X, or by any other value ν. Four values of ν were used in the previous chapter for the computation of λ_2.

For all practical purposes and for a relatively small value of X, $\Lambda_2(o,x)$ of eq. (6.1) is $\lambda_2 X$. For a given time position, λ_2 can be considered a constant. Therefore, Λ_2 is proportional to X. In that case, eq. (2.52) is a gamma distribution. The best estimate of λ_2, as discussed in Chapter V, should be used in computing the integral of eq. (6.1) for a given time position within the year.

The other alternative in computing λ_2 is to always use ν storms, with ν a number close to η_x, whenever the probability density function of a given X_ν is investigated as in Figs.5.1 through 5.7 This second alternative is used for the comparison of the theoretical probability density functions of X_ν, given by eq. (2.52), with the empirical frequency density curves of the four precipitation series, described in Chapter III.

6.2 Computations of probability density functions of storm precipitation. Distributions of the total precipitation amounts X_ν, for ν = 1, 2, 3 and 15, are studied for storms defined as each rainy day, by using eq. (2.52) and λ_2 values of the three examples (Durango, Fort Collins and Austin) as given by Figs. 5.1, 5.3 and 5.5. These investigations are carried out for the four time positions of the year: (1) January 1; (2) April 1; (3) July 1, and (4) October 1, as the representative dates of four annual seasons. In summary, for the first definition of storms, for the three examples of series of daily precipitation at Durango, Fort Collins and Austin, and for the above four seasons of the year, the theoretical probability density functions are determined for X_ν, with four cases ν = 1, 2, 3 and 15. The λ_2 values used are the computed values of λ_2, of Figs. 5.1, 5.3, and 5.5, and not the values of the fitted periodic component.

For a stronger test of how the theoretical probability density functions fit the frequency density curves, the use of λ_2 values from the fitted periodic components should be a better approach to use because it partly avoids the sampling fluctuations in λ_2.

The above four time positions are based on the assumption that the η_ν storms, which produce the total precipitation amount X_ν, are centered around the first day of January, April, July and October, respectively. The values λ_2 at those dates are used in the computation of $a_\nu(x)$ of eq. (2.52) which is the probability density function of X_ν.

Similarly as for the total precipitation amount X_ν of rainy days, with each rainy day considered as a storm, the probablity density functions are determined for X_ν of ν storms, with storms defined as uninterrupted sequences of rainy days, or of rainy hours in the case of Ames Station, with ν = 1, 2, 3 and 10, and for January 1, April 1, July 1 and October 1, respectively. These four dates represent in general lines the four annual seasons.

6.3 Computations of frequency density curves of storm precipitation. Frequency density curves of the total precipitation X_ν are determined for the following cases:

(1) For rainy days, each considered as a storm, for ν = 1, 2, 3 and 15, for the above four time positions, designated as Season-1, Season-2, Season-3 and Season-4 in all figures, and for the three examples of Durango, Fort Collins and Austin daily precipitation series.

(2) For storms, defined as uninterrupted sequences of rainy days or rainy hours, whichever is relevant, for ν = 1, 2, 3 and 10, for the same above four time positions, and for the four examples of the Durango, Fort Collins, Austin and Ames precipitation series.

For the first definition of storms, Figs. 6.1, 6.3, and 6.5 each give these 16 empirical frequency density curves of X_ν (the four values of ν = 1, 2, 3, 15, and each of them for the four seasons). For the second definition of storms, Figs. 6.2, 6.4, 6.6, and 6.7, each give these 16 frequency density curves of X_ν (the four values of ν = 1, 2, 3, 10, and each of them for the four seasons).

6.4 Comparison of theoretical probability density functions with the empirical frequency density curves of storm precipitation. This comparison is given in Figs. 6.1 through 6.7, in which the solid lines refer to the computed frequency density curves, while dashed lines refer to the theoretical probability density functions.

It is usually customary to compare the theoretical probability distributions with the empirical cummulative

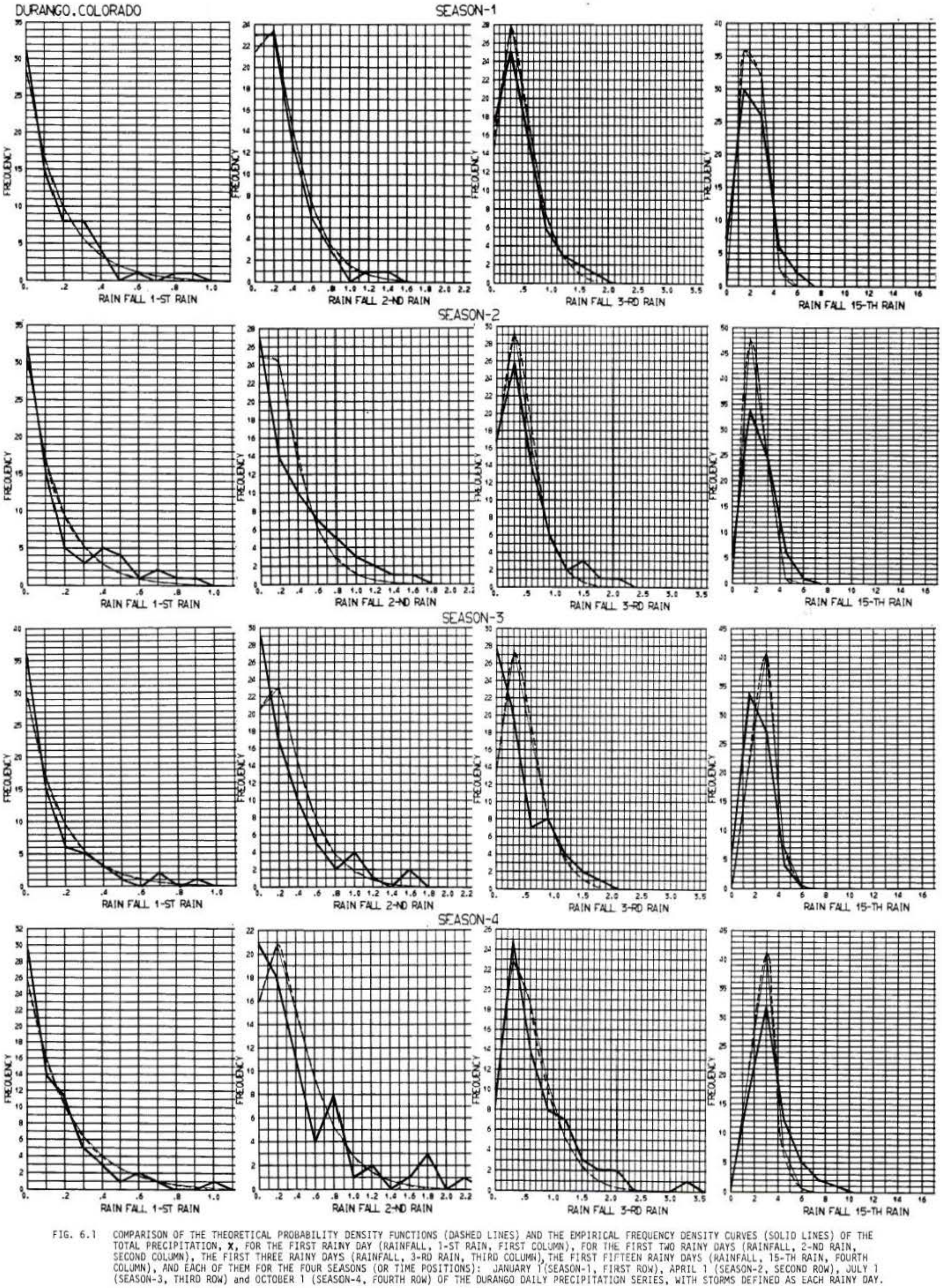

FIG. 6.1 COMPARISON OF THE THEORETICAL PROBABILITY DENSITY FUNCTIONS (DASHED LINES) AND THE EMPIRICAL FREQUENCY DENSITY CURVES (SOLID LINES) OF THE TOTAL PRECIPITATION, **x**, FOR THE FIRST RAINY DAY (RAINFALL, 1-ST RAIN, FIRST COLUMN), FOR THE FIRST TWO RAINY DAYS (RAINFALL, 2-ND RAIN, SECOND COLUMN), THE FIRST THREE RAINY DAYS (RAINFALL, 3-RD RAIN, THIRD COLUMN), THE FIRST FIFTEEN RAINY DAYS (RAINFALL, 15-TH RAIN, FOURTH COLUMN), AND EACH OF THEM FOR THE FOUR SEASONS (OR TIME POSITIONS): JANUARY 1 (SEASON-1, FIRST ROW), APRIL 1 (SEASON-2, SECOND ROW), JULY 1 (SEASON-3, THIRD ROW) and OCTOBER 1 (SEASON-4, FOURTH ROW) OF THE DURANGO DAILY PRECIPITATION SERIES, WITH STORMS DEFINED AS EACH RAINY DAY.

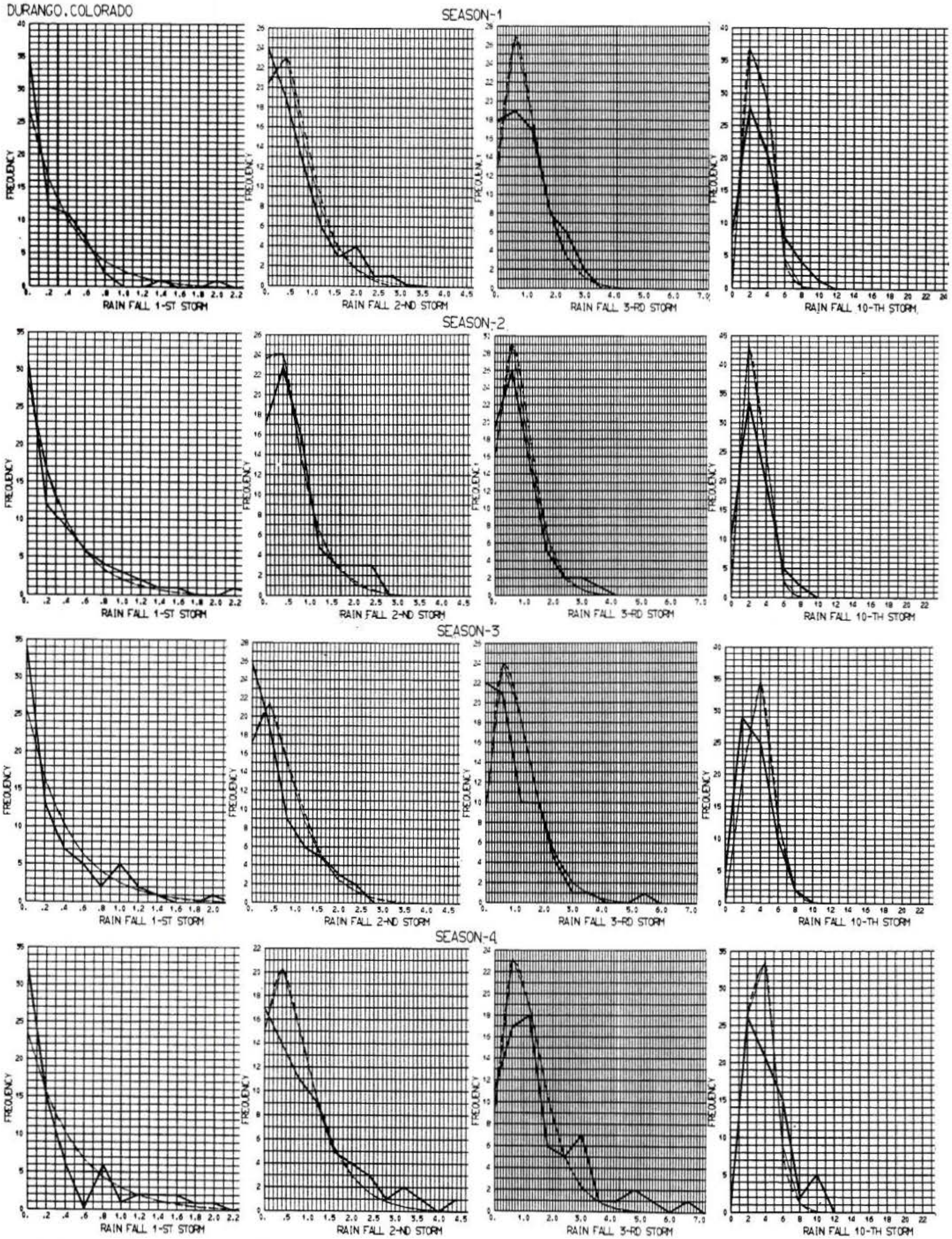

FIG. 6.2 COMPARISON OF THE THEORETICAL PROBABILITY DENSITY FUNCTIONS (DASHED LINES) AND THE EMPIRICAL FREQUENCY DENSITY CURVES (SOLID LINES) OF THE TOTAL PRECIPITATION, **X**, FOR THE FIRST STORM (RAINFALL, 1-ST STORM, FIRST COLUMN), THE FIRST TWO STORMS (RAINFALL, 2-ND STORM, SECOND COLUMN), THE FIRST THREE STORMS (RAINFALL, 3-RD STORM, THIRD COLUMN), THE FIRST TEN STORMS (RAINFALL, 10-TH STORM, FOURTH COLUMN), AND EACH OF THEM FOR THE FOUR SEASONS (OR TIME POSITIONS): JANUARY 1 (SEASON-1, FIRST ROW), APRIL 1 (SEASON-2, SECOND ROW), JULY 1 (SEASON-3, THIRD ROW), AND OCTOBER 1 (SEASON-4, FOURTH ROW) OF THE DURANGO DAILY PRECIPITATION SERIES, WITH STORMS DEFINED AS UNINTERRUPTED SEQUENCES OF RAINY DAYS.

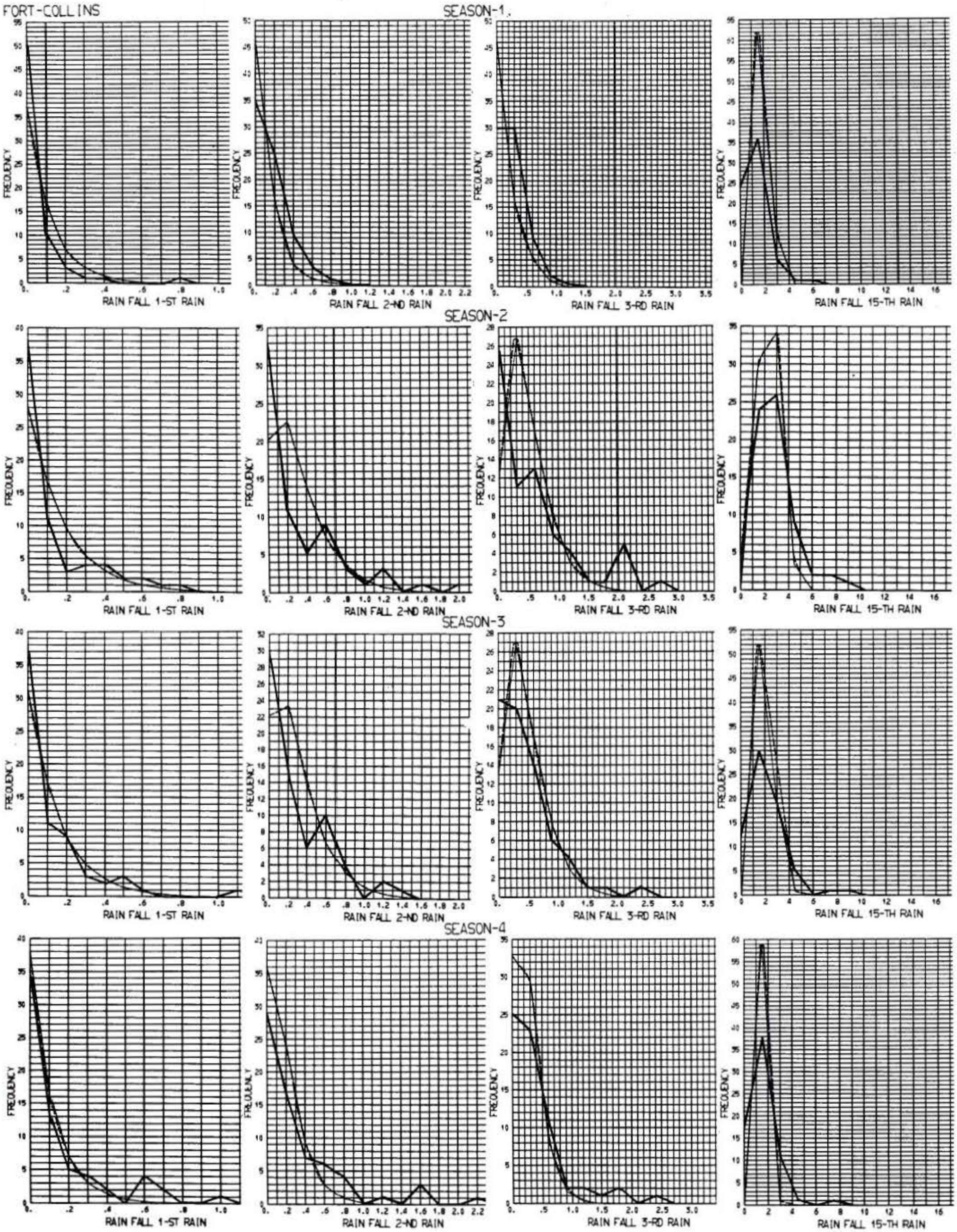

FIG. 6.3 COMPARISON OF THE THEORETICAL PROBABILITY DENSITY FUNCTIONS (DASHED LINES) AND THE EMPIRICAL FREQUENCY DENSITY CURVES (SOLID LINES) OF THE TOTAL PRECIPITATION, **X**, FOR THE FIRST RAINY DAY (RAINFALL, 1-ST RAIN, FIRST COLUMN), FOR THE FIRST TWO RAINY DAYS (RAINFALL, 2-ND RAIN, SECOND COLUMN), THE FIRST THREE RAINY DAYS (RAINFALL, 3-RD RAIN, THIRD COLUMN), THE FIRST FIFTEEN RAINY DAYS (RAINFALL, 15-TH RAIN, FOURTH COLUMN), AND EACH OF THEM FOR THE FOUR SEASONS (OR TIME POSITIONS): JANUARY 1 (SEASON-1, FIRST ROW), APRIL 1 (SEASON-2, SECOND ROW), JULY 1 (SEASON-3, THIRD ROW) AND OCTOBER 1 (SEASON-4, FOURTH ROW) OF THE FORT COLLINS DAILY PRECIPITATION SERIES, WITH STORMS DEFINED AS EACH RAINY DAY.

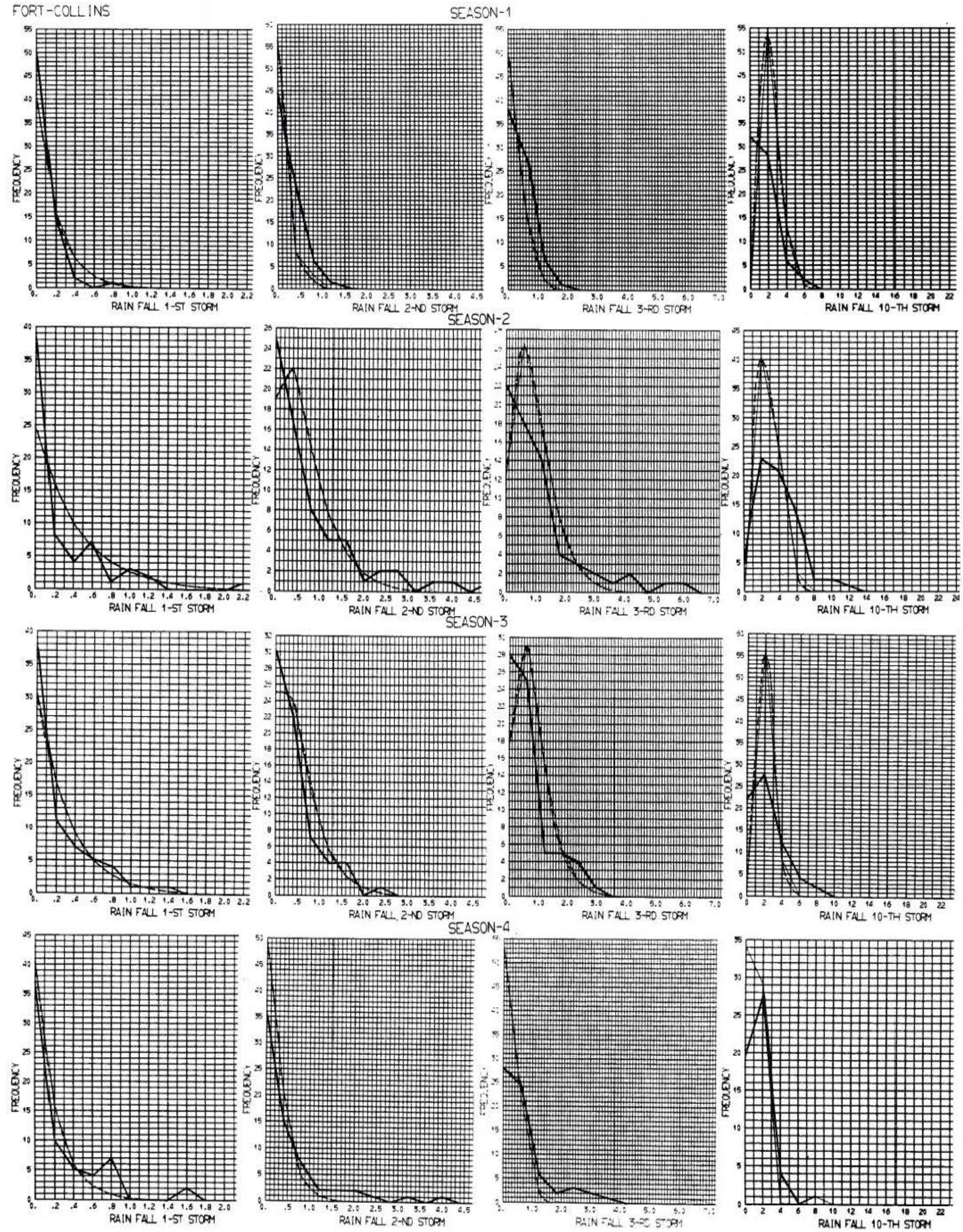

FIG. 6.4 COMPARISON OF THE THEORETICAL PROBABILITY DENSITY FUNCTIONS (DASHED LINES) AND THE EMPIRICAL FREQUENCY DENSITY CURVES (SOLID LINES) OF THE TOTAL PRECIPITATION, **X**, FOR THE FIRST STORM (RAINFALL, 1-ST STORM, FIRST COLUMN), THE FIRST TWO STORMS (RAINFALL, 2-ND STORM, SECOND COLUMN), THE FIRST THREE STORMS (RAINFALL, 3-RD STORM, THIRD COLUMN), THE FIRST TEN STORMS (RAINFALL, 10-TH STORM, FOURTH COLUMN), AND EACH OF THEM FOR THE FOUR SEASONS (OR TIME POSITIONS): JANUARY 1 (SEASON-1, FIRST ROW), APRIL 1 (SEASON-2, SECOND ROW), JULY 1 (SEASON-3, THIRD ROW), AND OCTOBER 1 (SEASON-4, FOURTH ROW) OF THE FORT COLLINS DAILY PRECIPITATION SERIES, WITH STORMS DEFINED AS UNINTERRUPTED SEQUENCES OF RAINY DAYS.

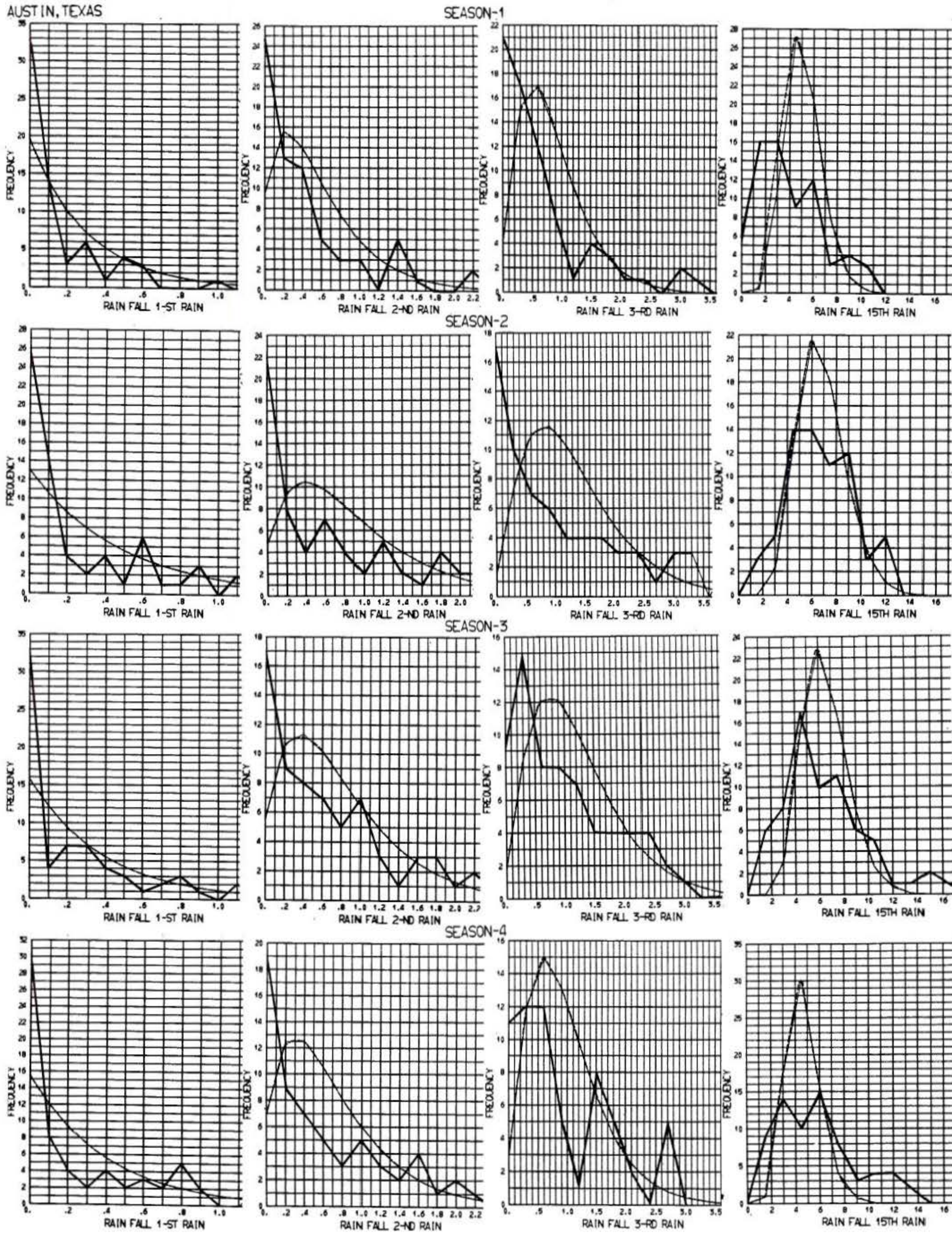

FIG. 6.5 COMPARISON OF THE THEORETICAL PROBABILITY DENSITY FUNCTIONS (DASHED LINES) AND THE EMPIRICAL FREQUENCY DENSITY CURVES (SOLID LINES) OF THE TOTAL PRECIPITATION, **X**, FOR THE FIRST RAINY DAY (RAINFALL, 1-ST RAIN, FIRST COLUMN), FOR THE FIRST TWO RAINY DAYS (RAINFALL, 2-ND RAIN, SECOND COLUMN), THE FIRST THREE RAINY DAYS (RAINFALL, 3-RD RAIN, THIRD COLUMN), THE FIRST FIFTEEN RAINY DAYS (RAINFALL, 15-TH RAIN, FOURTH COLUMN), AND EACH OF THEM FOR THE FOUR SEASONS (OR TIME POSITIONS): JANUARY 1 (SEASON-1, FIRST ROW), APRIL 1 (SEASON-2, SECOND ROW), JULY 1 (SEASON-3, THIRD ROW), AND OCTOBER 1 (SEASON-4, FOURTH ROW) OF THE AUSTIN DAILY PRECIPITATION SERIES, WITH STORMS DEFINED AS EACH RAINY DAY.

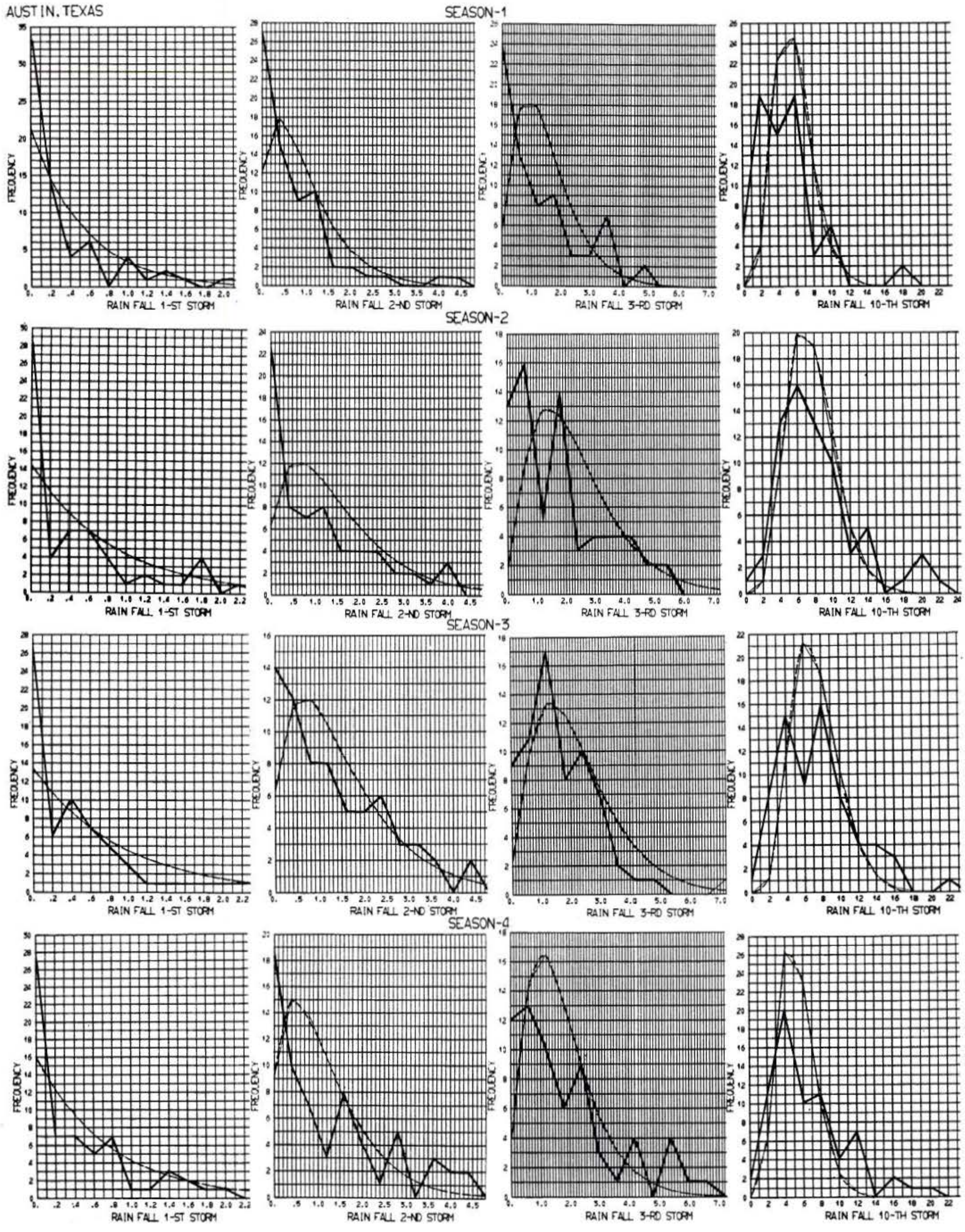

FIG. 6.6 COMPARISON OF THE THEORETICAL PROBABILITY DENSITY FUNCTIONS (DASHED LINES) AND THE EMPIRICAL FREQUENCY DENSITY CURVES (SOLID LINES) OF THE TOTAL PRECIPITATION, **X**, FOR THE FIRST STORM (RAINFALL, 1-ST STORM, FIRST COLUMN), THE FIRST TWO STORMS (RAINFALL, 2-ND STORM, SECOND COLUMN), THE FIRST THREE STORMS (RAINFALL, 3-RD STORM, THIRD COLUMN), THE FIRST TEN STORMS (RAINFALL, 10-TH STORM, FOURTH COLUMN), AND EACH OF THEM FOR THE FOUR SEASONS (OR TIME POSITIONS): JANUARY 1 (SEASON-1, FIRST ROW), APRIL 1 (SEASON-2, SECOND ROW), JULY 1 (SEASON-3, THIRD ROW), AND OCTOBER 1 (SEASON-4, FOURTH ROW) OF THE AUSTIN DAILY PRECIPITATION SERIES, WITH STORMS DEFINED AS UNINTERRUPTED SEQUENCES OF RAINY DAYS.

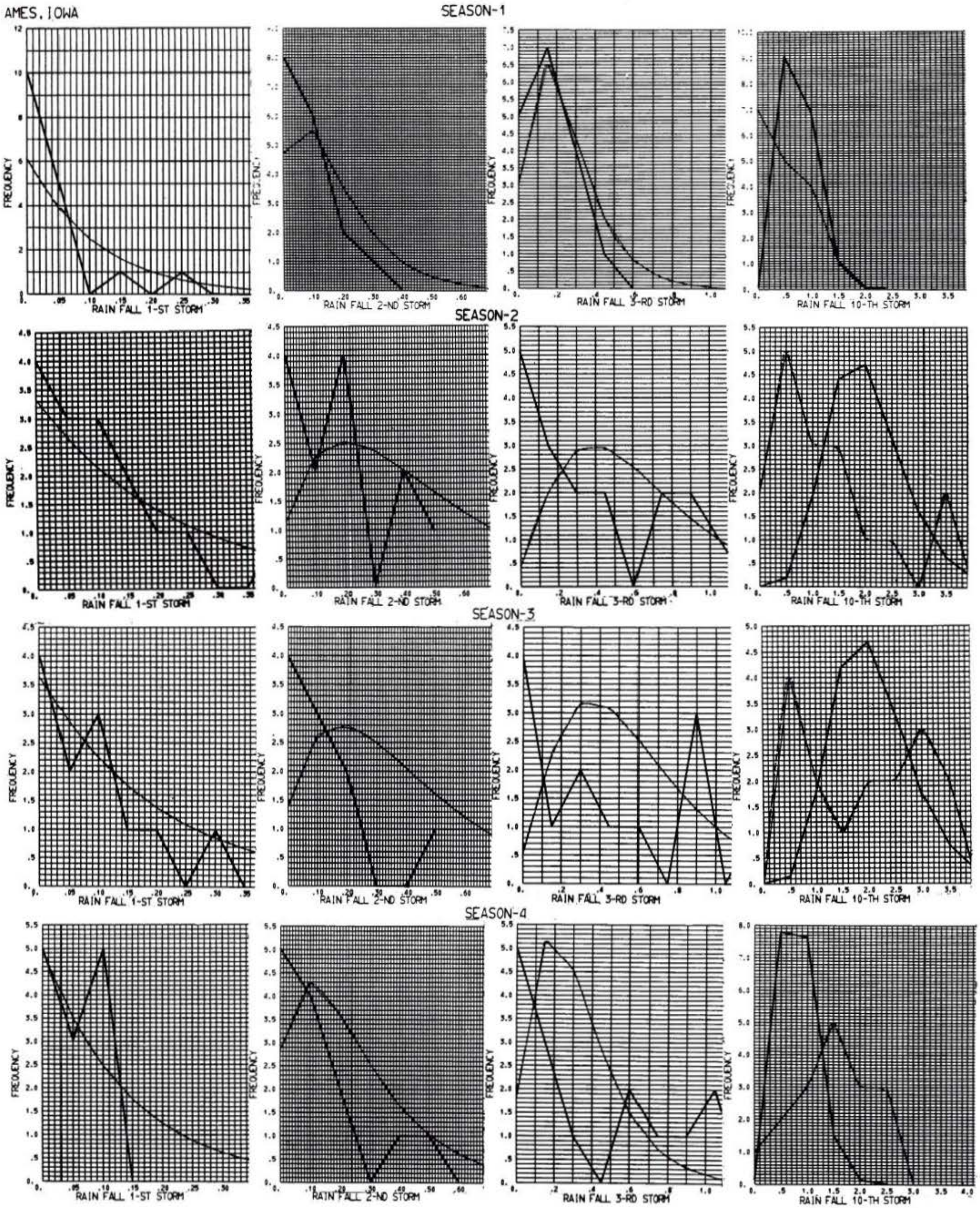

FIG. 6.7 COMPARISON OF THE THEORETICAL PROBABILITY DENSITY FUNCTIONS (DASHED LINES) AND THE EMPIRICAL FREQUENCY DENSITY CURVES (SOLID LINES) OF THE TOTAL PRECIPITATION, **X**, FOR THE FIRST STORM (RAINFALL, 1-ST STORM, FIRST COLUMN), THE FIRST TWO STORMS (RAINFALL, 2-ND STORM, SECOND COLUMN), THE FIRST THREE STORMS (RAINFALL, 3-RD STORM, THIRD COLUMN), THE FIRST TEN STORMS (RAINFALL, 10-TH STORM, FOURTH COLUMN), AND EACH OF THEM FOR THE FOUR SEASONS (OR TIME POSITIONS): JANUARY 1 (SEASON-1, FIRST ROW), APRIL 1 (SEASON-2, SECOND ROW), JULY 1 (SEASON-3, THIRD ROW), AND OCTOBER 1 (SEASON-4, FOURTH ROW) OF THE AMES HOURLY PRECIPITATION SERIES, WITH STORMS DEFINED AS UNINTERRUPTED SEQUENCES OF RAINY HOURS.

frequency distributions, rather than to compare the density functions with the frequency density curves. The fact is that the comparison of distribution looks better to an eye, than the comparison of their density curves. This can be seen best by comparing the graphs of figures in this chapter, in which case the density curves are compared with the graphs of figures in the following chapters where distributions are compared.

The eye inference is often misleading and unavoidably represents a subjective decision. A comparison of two curves, one theoretical and another empirical, may look as good to one person and very bad to another. The objective statistical inference by using the parameters and tests of hypotheses is the only proper way of comparing theoretical and empirical curves. Because readers are often accustomed to drawing their own conclusions, the graphical presentation is given in this paper, rather than the tables of chi-square or any other statistic, as the results of statistical inference tests. As mentioned in Chapter III, this is not an exhaustive study of a large number of stations, with a statistical analysis performed of how the various theoretical distribution functions of precipitation stochastic process fit the empirical distributions. It is rather a development of methodology, with the four examples used to illustrate its potential usefulness.

Figures 6.1 and 6.2 refer to Durango, Fig. 6.3 and 6.4 to Fort Collins, and Figs. 6.5 and 6.6 to Austin daily precipitation series for each of the two definitions of storms, respectively. As the number of years of data is 71 for Durango, 69 for Fort Collins, and 70 for Austin, those are also the sample sizes of the empirical frequency density curves (solid lines) of Figs. 6.1 - 6.6. Differences between the theoretical probability densities and the empirical frequency densities are smaller for Durango and Fort Collins than for Austin. Figure 6.7 refers to Ames hourly precipitation series. The sample size is 18 years only. This small sample explains why differences between the theoretical probability densities and the empirical frequency densities are much greater for this station than for the other three. The use of hourly data in the case of the Ames Station may further explain why these differences are so large.

In summary, the gamma distribution of eq. (2.52), with $\Lambda_2(o,x)$ of eq. (6.1) assumed to be proportional to X for a given λ_2 and a given position in time, seems to well fit the empirical distributions of the storm precipitation amounts.

Chapter VII

PROBABILITY DISTRIBUTIONS OF TIME OCCURRENCE OF STORMS

7.1 Definition of time occurrence of storms. If a storm is defined as a rainy day or a rainy hour, the lapse time from the beginning of an interval to that rainy day or rainy hour is its time occurrence. If a storm is defined as uninterrupted sequence of rainy days or rainy hours, then the lapse time from the beginning of an interval to the last rainy day or last rainy hour of that sequence is its time occurrence. If t_o stands for the time when the observation begins, then τ_ν denotes the end of ν-th storm. The difference $\tau_\nu - t_o$ is called the lapse time of the ν-th storm. For $t_o = o$, τ_ν is the lapse time.

The probability density function of τ_ν is given by eq. (2.48). The following integral represents the average number of lapsed times in $(t_o, t]$

$$\Lambda_1(t_o,t) = \int_{t_o}^{t} \lambda_1(s)ds \tag{7.1}$$

and it is $\lambda_1(t-t_o)$ only if λ_1 is a constant with time.

7.2 Computation of theoretical probability distributions of lapse time, τ_ν. Assuming that t_o is defined by dates of the year, in the examples of this study, four values of t_o are taken as four seasons: January 1 (Season-1), April 1 (Season-2), July 1 (Season-3) and October 1 (Season-4). Then for any value $\tau_\nu = t$, the various terms of eq. 2.48 are determined from λ_1-time functions, as computed for four examples and presented in Chapter IV. The computed λ_1 values are used rather than their values from the fitted periodic components. This latter case should be used whenever some sampling variations in λ_1 should be avoided.

The term $\lambda_1(t)$ in eq. (2.48) is the computed $_1$ value at the time t. The term in eq. (2.48), given by eq. (7.1), is the integral from the beginning of the above dates and various values of t. This procedure is used for $\nu = 1, 2, 3$ and 15 for the first definition and for $\nu = 1, 2, 3$ and 10 for the second definition of storms.

The theoretical probability distribution functions (as integrals of eq. (2.48)) are given for the four examples of precipitation data (Durango, Fort Collins, Austin and Ames) as light solid lines in Figs. 7.1 through 7.7 as explained in the captions of these figures.

7.3 Computation of empirical frequency distributions of lapse time, τ_ν. Frequency distributions of lapse time τ_ν, given in all figures as heavy solid lines, are determined for the following cases:

(1) For rainy days, each considered as a storm, for $\nu = 1, 2, 3$ and 15, and for the above four time positions t_o, designated as Season-1, Season-2, Season-3, and Season-6 in all figures, and for the three examples of Durango, Fort Collins, and Austin daily precipitation series.

(2) For storms, defined as uninterrupted sequences of rainy days or rainy hours, whichever is relevant, for $\nu = 1, 2, 3$ and 10, and for the same above four positions of t_o, and for the four examples of Durango, Fort Collins, Austin, and Ames precipitation series.

For the first definition of storms, Figs. 7.1, 7.3, and 7.5 each give these 16 empirical frequency distributions of τ_ν (the four values of $\nu = 1, 2, 3$ and 15, and each of them for the four seasons). For the second definition of storms, Figs. 7.2, 7.4, 7.6, and 7.7 each give these 16 empirical frequency distributions of τ_ν (the four values of $\nu = 1, 2, 3$ and 10, and each of them for the four seasons).

7.4 Comparison of theoretical probability distributions and empirical frequency distributions of lapse time, τ_ν. This comparison in Figs. 7.1 through 7.7 shows a good closeness of theoretical and empirical curves, though this is not studied by tests of appropriate statistics of goodnesss of fit. The exception is Fig. 7.7, for the hourly precipitation data at Ames, because the sample is small, only 18 years.

In conclusion, eq. (2.48) gives a good distribution of τ_ν, provided λ_1 is well estimated as a function of time. The values of λ_1 from the periodic component should be used instead of the computed values of λ_1.

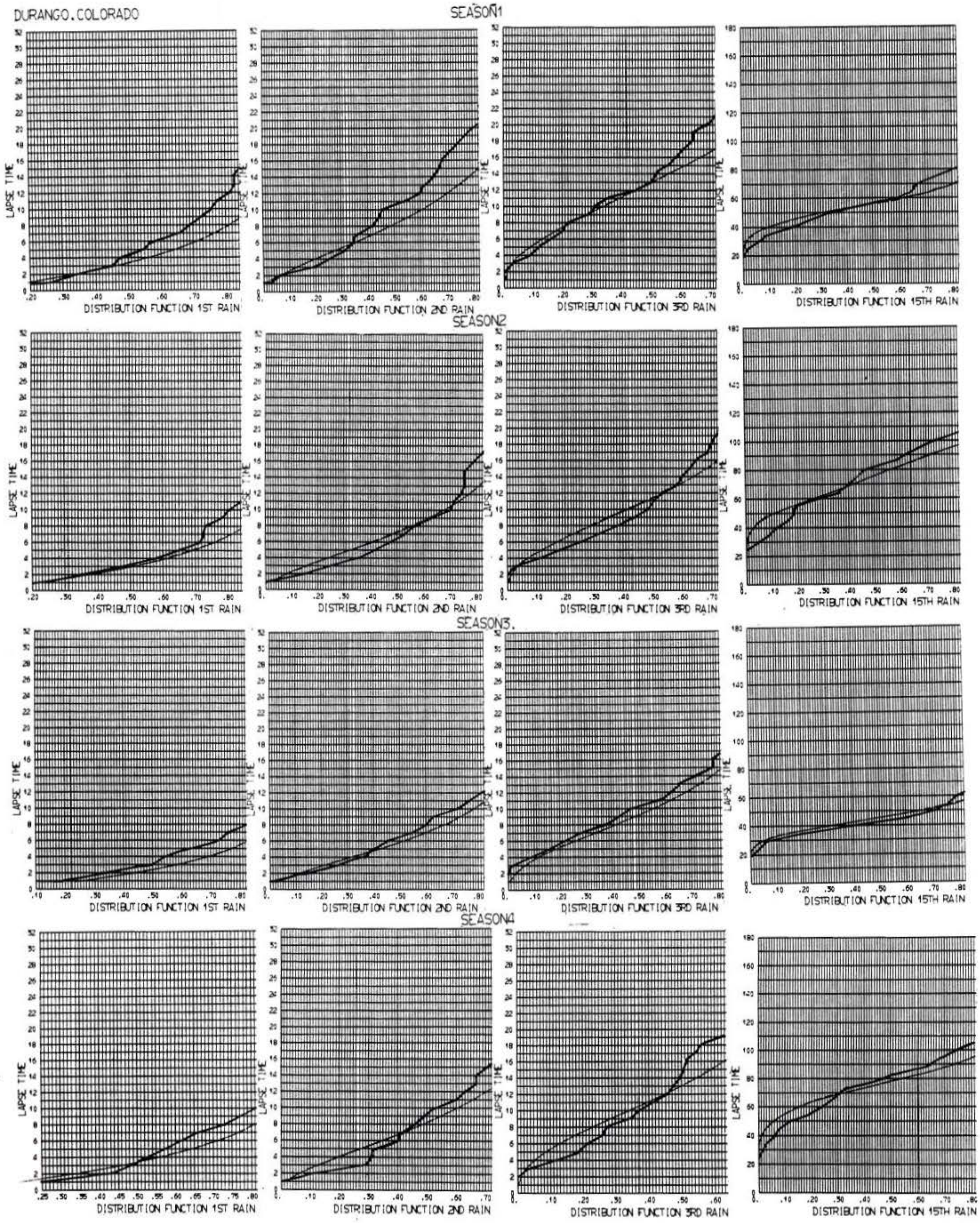

FIG. 7.1 COMPARISON OF THE THEORETICAL PROBABILITY DISTRIBUTION (LIGHT SOLID LINES) AND THE EMPIRICAL FREQUENCY DISTRIBUTIONS (HEAVY SOLID LINES) OF THE STORM LAPSE TIME, τ_{ij}, FOR THE FIRST RAINY DAY (RAINFALL, 1-ST RAIN, FIRST COLUMN), FOR THE FIRST TWO RAINY DAYS (RAINFALL, 2-ND RAIN, SECOND COLUMN), FOR THE FIRST THREE RAINY DAYS (RAINFALL, 3-RD RAIN, THIRD COLUMN), AND FOR THE FIRST FIFTEEN RAINY DAYS (RAINFALL, 15-TH RAIN, FOURTH COLUMN), AND EACH OF THEM FOR THE FOUR SEASONS (OR TIME POSITIONS): JANUARY 1 (SEASON-1, FIRST ROW), APRIL 1 (SEASON-2, SECOND ROW), JULY 1 (SEASON-3, THIRD ROW), AND OCTOBER 1 (SEASON-4, FOURTH ROW) OF DURANGO DAILY PRECIPITATION SERIES, WITH STORMS DEFINED AS EACH RAINY DAY.

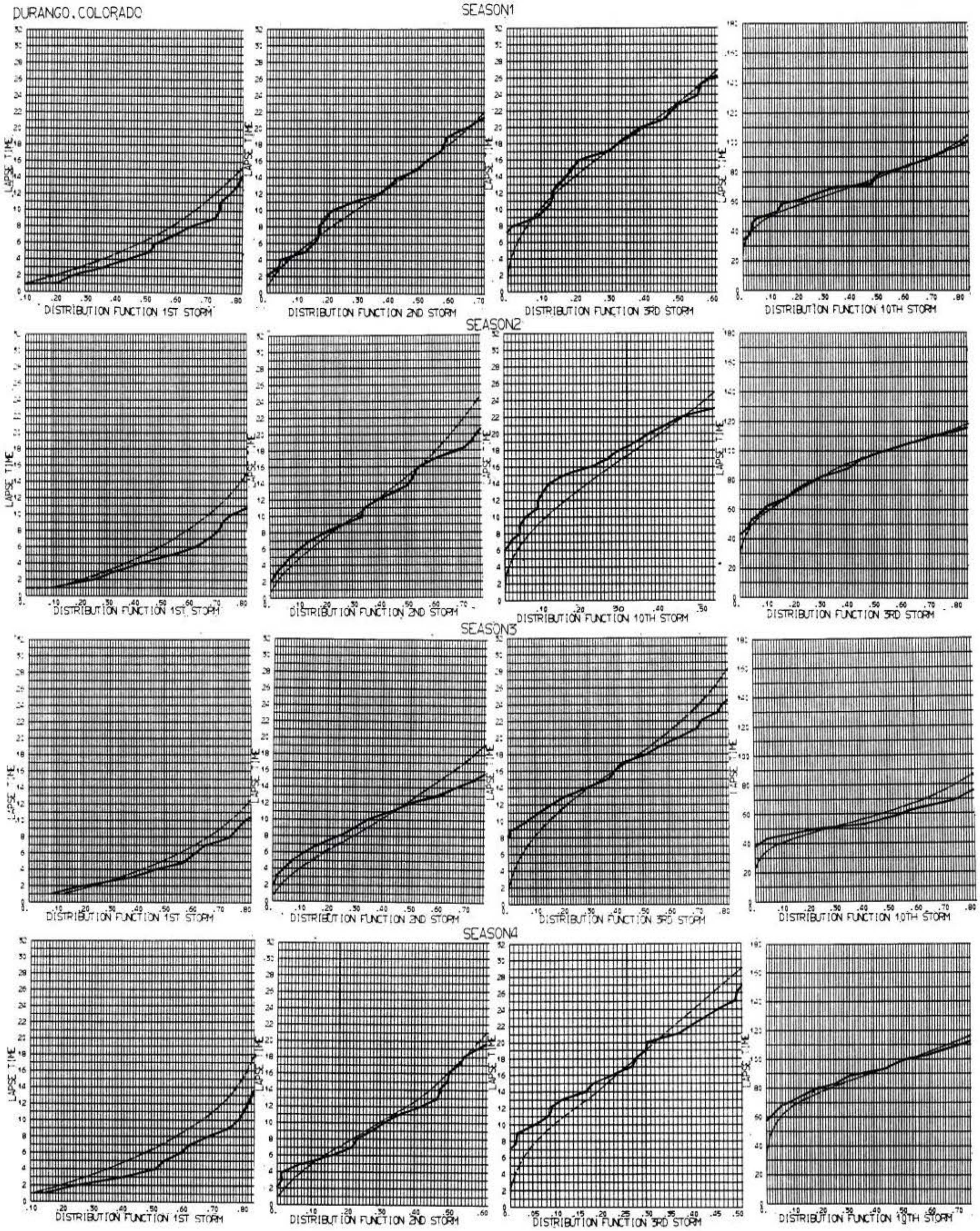

FIG. 7.2 COMPARISON OF THE THEORETICAL PROBABILITY DISTRIBUTION (LIGHT SOLID LINES) AND THE EMPIRICAL FREQUENCY DISTRIBUTIONS (HEAVY SOLID LINES) OF THE STORM LAPSE TIME, τ_υ, FOR THE FIRST STORM (RAINFALL, 1-ST STORM, FIRST COLUMN), FOR THE FIRST TWO STORMS (RAINFALL, 2-ND STORM, SECOND COLUMN), FOR THE FIRST THREE STORMS (RAINFALL, 3-RD STORM, THIRD COLUMN), AND THE FIRST TEN STORMS (RAINFALL, 10-TH STORM, FOURTH COLUMN), AND EACH OF THEM FOR THE FOUR SEASONS (OR TIME POSITIONS): JANUARY 1 (SEASON-1, FIRST ROW), APRIL 1 (SEASON-2, SECOND ROW), JULY 1 (SEASON-3, THIRD ROW), AND OCTOBER 1 (SEASON-4, FOURTH ROW) OF DURANGO DAILY PRECIPITATION SERIES, WITH STORMS DEFINED AS UNINTERRUPTED SEQUENCES OF RAINY DAYS.

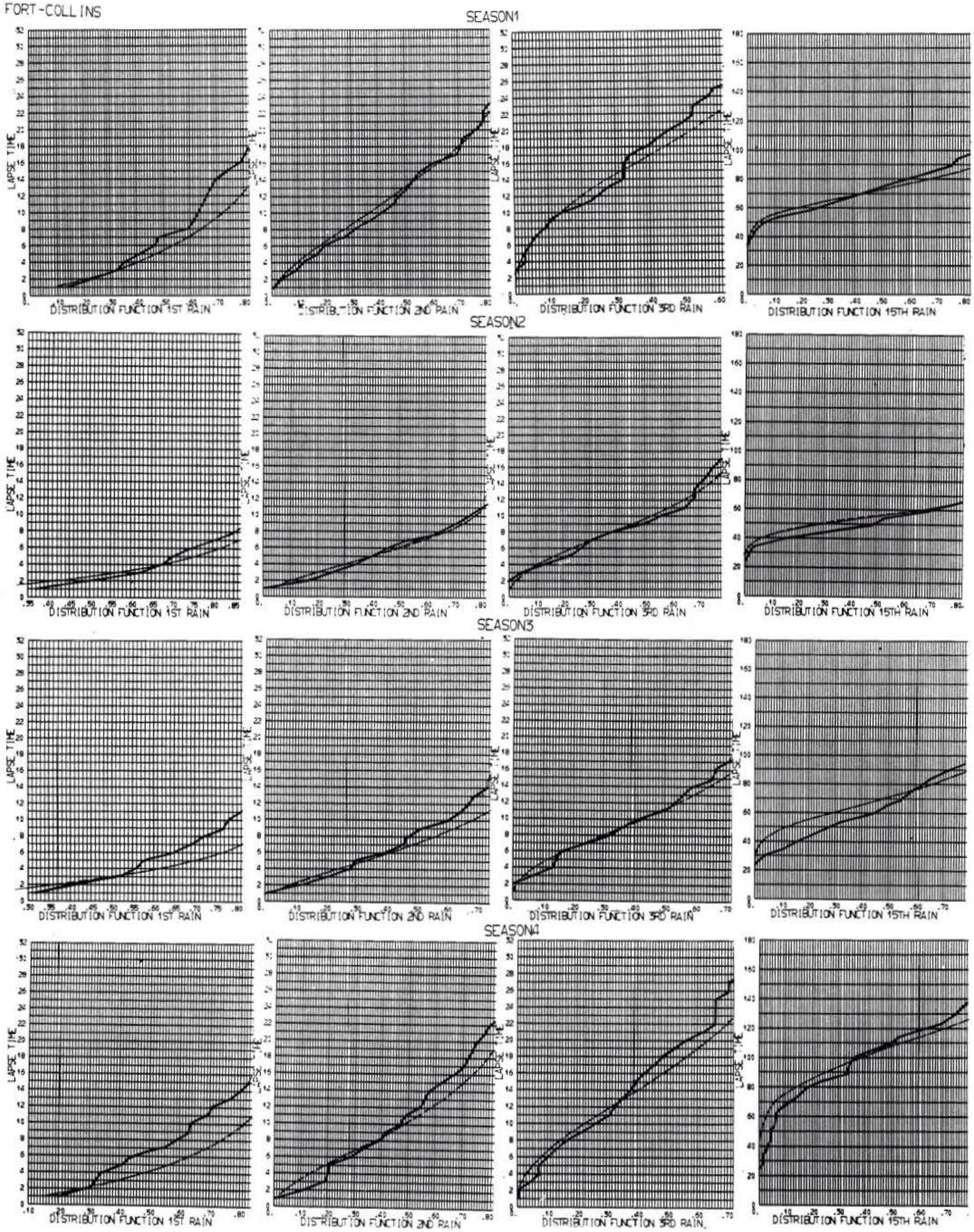

FIG. 7.3 COMPARISON OF THE THEORETICAL PROBABILITY DISTRIBUTION (LIGHT SOLID LINES) AND THE EMPIRICAL FREQUENCY DISTRIBUTIONS (HEAVY SOLID LINES) OF THE STORM LAPSE TIME, τ_v, FOR THE FIRST RAINY DAY (RAINFALL, 1-ST RAIN, FIRST COLUMN), FOR THE FIRST TWO RAINY DAYS (RAINFALL, 2-ND RAIN, SECOND COLUMN), FOR THE FIRST THREE RAINY DAYS (RAINFALL, 3-RD RAIN, THIRD COLUMN), AND FOR THE FIRST FIFTEEN RAINY DAYS (RAINFALL, 15-TH RAIN, FOURTH COLUMN), AND EACH OF THEM FOR THE FOUR SEASONS (OR TIME POSITIONS): JANUARY 1 (SEASON-1, FIRST ROW), APRIL 1 (SEASON-2, SECOND ROW), JULY 1 (SEASON-3, THIRD ROW), AND OCTOBER 1 (SEASON-4, FOURTH ROW) OF FORT COLLINS DAILY PRECIPITATION SERIES, WITH STORMS DEFINED AS EACH RAINY DAY.

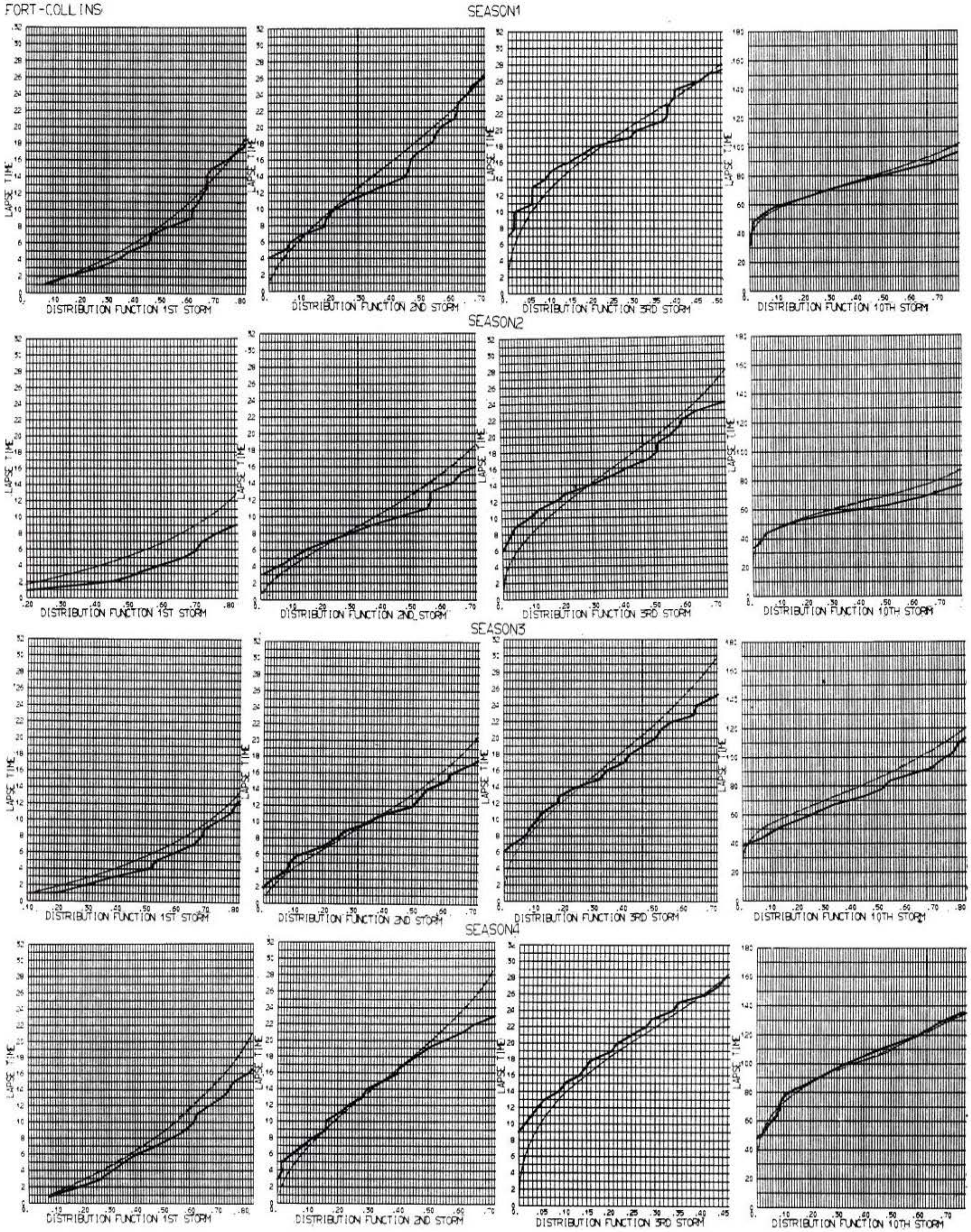

FIG. 7.4 COMPARISON OF THE THEORETICAL PROBABILITY DISTRIBUTION (LIGHT SOLID LINES) AND THE EMPIRICAL FREQUENCY DISTRIBUTIONS (HEAVY SOLID LINES) OF THE STORM LAPSE TIME, τ_ν, FOR THE FIRST STORM (RAINFALL, 1-ST STORM, FIRST COLUMN), FOR THE FIRST TWO STORMS (RAINFALL, 2-ND STORM, SECOND COLUMN), FOR THE FIRST THREE STORMS (RAINFALL, 3-RD STORM, THIRD COLUMN), AND THE FIRST TEN STORMS (RAINFALL, 10-TH STORM, FOURTH COLUMN), AND EACH OF THEM FOR THE FOUR SEASONS (OR TIME POSITIONS): JANUARY 1 (SEASON-1. FIRST ROW), APRIL 1 (SEASON-2, SECOND ROW), JULY 1 (SEASON-3, THIRD ROW), AND OCTOBER 1 (SEASON-4, FOURTH ROW) OF FORT COLLINS DAILY PRECIPITATION SERIES, WITH STORMS DEFINED AS UNINTERRUPTED SEQUENCES OF RAINY DAYS.

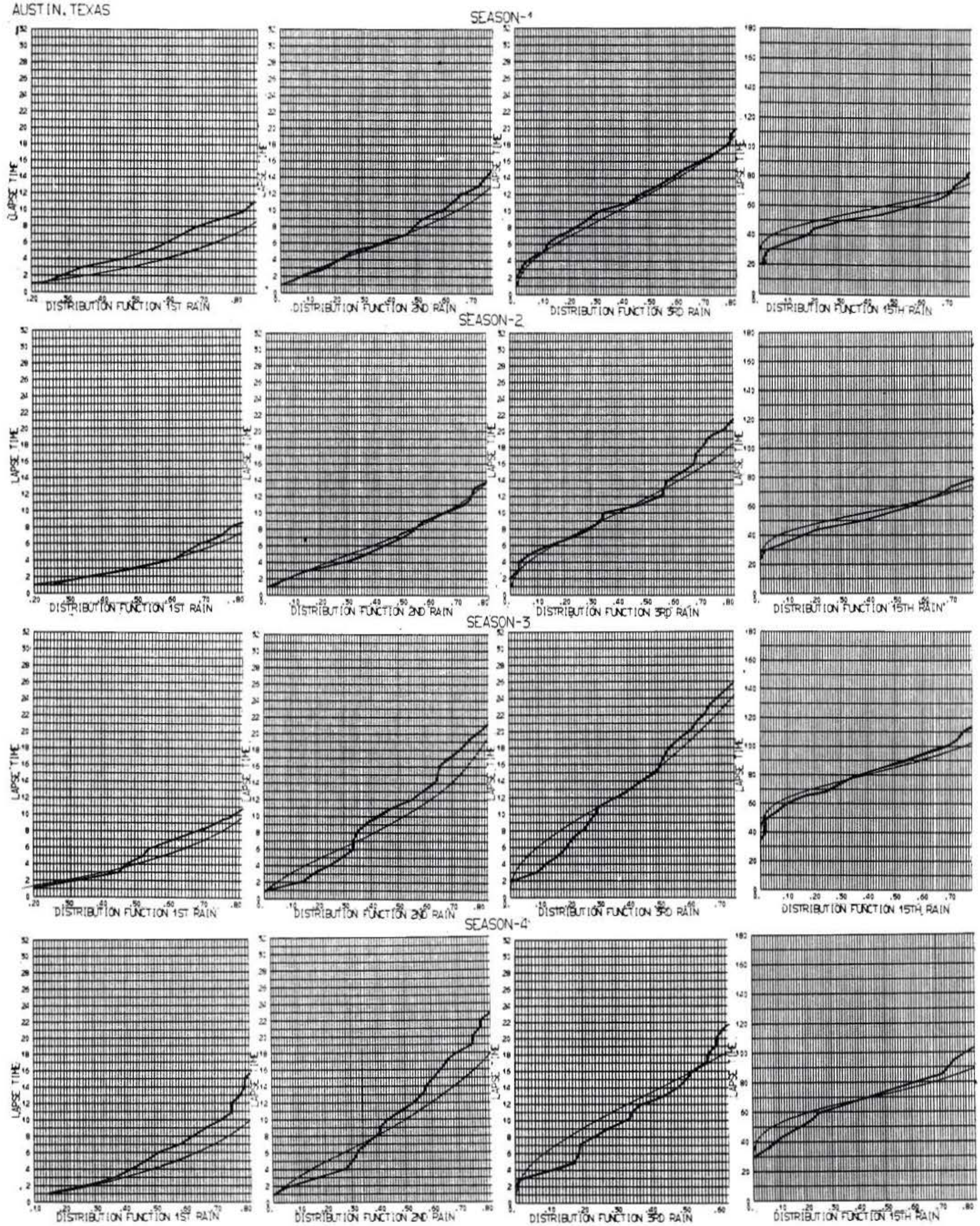

FIG. 7.5 COMPARISON OF THE THEORETICAL PROBABILITY DISTRIBUTION (LIGHT SOLID LINES) AND THE EMPIRICAL FREQUENCY DISTRIBUTIONS (HEAVY SOLID LINES) OF THE STORM LAPSE TIME, τ_ν, FOR THE FIRST RAINY DAY (RAINFALL, 1-ST RAIN, FIRST COLUMN), FOR THE FIRST TWO RAINY DAYS (RAINFALL, 2-ND RAIN, SECOND COLUMN), FOR THE FIRST THREE RAINY DAYS (RAINFALL, 3-RD RAIN, THIRD COLUMN), AND FOR THE FIRST FIFTEEN RAINY DAYS (RAINFALL, 15-TH RAIN, FOURTH COLUMN), AND EACH OF THEM FOR THE FOUR SEASONS (OR TIME POSITIONS): JANUARY 1 (SEASON-1, FIRST ROW), APRIL 1 (SEASON-2, SECOND ROW), JULY 1 (SEASON-3, THIRD ROW), AND OCTOBER 1 (SEASON-4, FOURTH ROW) OF AUSTIN DAILY PRECIPITATION SERIES, WITH STORMS DEFINED AS EACH RAINY DAY.

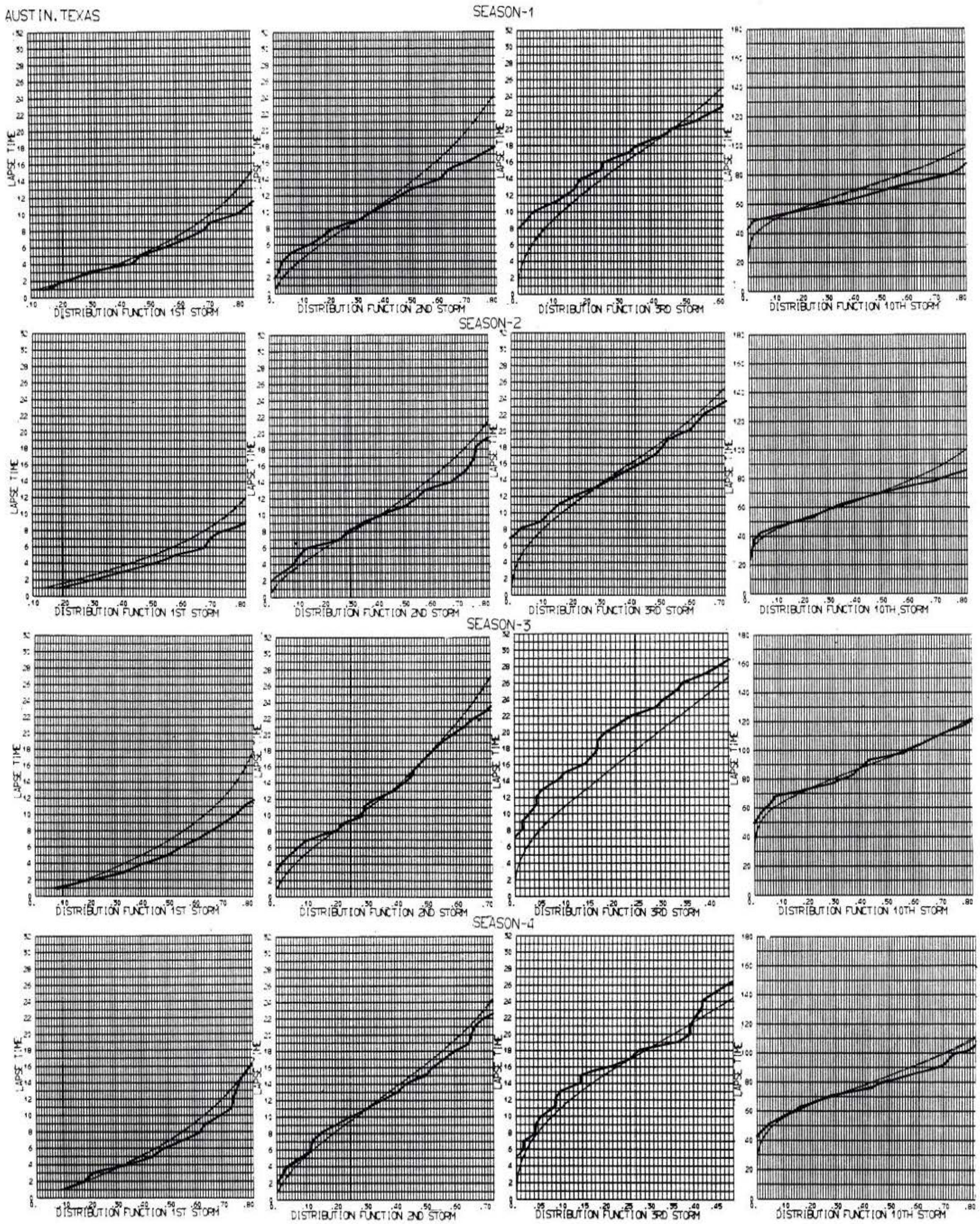

FIG. 7.6 COMPARISON OF THE THEORETICAL PROBABILITY DISTRIBUTION (LIGHT SOLID LINES) AND THE EMPIRICAL FREQUENCY DISTRIBUTIONS (HEAVY SOLID LINES) OF THE STORM LAPSE TIME, τ_v, FOR THE FIRST STORM (RAINFALL, 1-ST STORM, FIRST COLUMN), FOR THE FIRST TWO STORMS (RAINFALL, 2-ND STORM, SECOND COLUMN), FOR THE FIRST THREE STORMS (RAINFALL, 3-RD STORM, THIRD COLUMN), AND THE FIRST TEN STORMS (RAINFALL, 10-TH STORM, FOURTH COLUMN), AND EACH OF THEM FOR THE FOUR SEASONS (OR TIME POSITIONS): JANUARY 1 (SEASON-1, FIRST ROW), APRIL 1 (SEASON-2, SECOND ROW), JULY 1 (SEASON-3, THIRD ROW), AND OCTOBER 1 (SEASON-4, FOURTH ROW) OF AUSTIN DAILY PRECIPITATION SERIES, WITH STORMS DEFINED AS UNINTERRUPTED SEQUENCES OF RAINY DAYS.

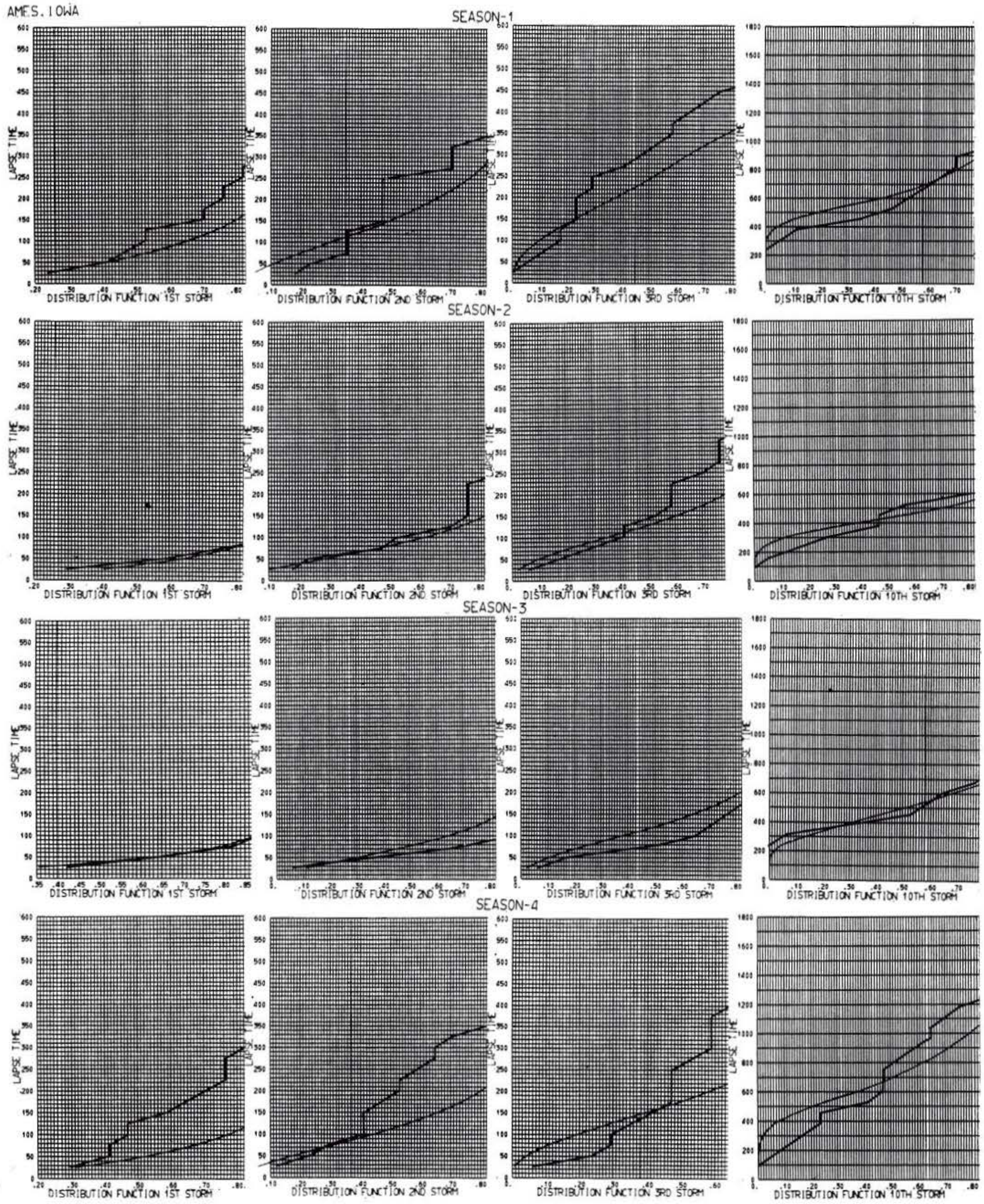

FIG. 7.7 COMPARISON OF THE THEORETICAL PROBABILITY DISTRIBUTION (LIGHT SOLID LINES) AND THE EMPIRICAL FREQUENCY DISTRIBUTIONS (HEAVY SOLID LINES) OF THE STORM LAPSE TIME, τ_v, FOR THE FIRST STORM (RAINFALL, 1-ST STORM, FIRST COLUMN), FOR THE FIRST TWO STORMS (RAINFALL, 2-ND STORM, SECOND COLUMN), FOR THE FIRST THREE STORMS (RAINFALL, 3-RD STORM, THIRD COLUMN), AND THE FIRST TEN STORMS (RAINFALL, 10-TH STORM, FOURTH COLUMN), AND EACH OF THEM FOR THE FOUR SEASONS (OR TIME POSITIONS): JANUARY 1 (SEASON-1, FIRST ROW), APRIL 1 (SEASON-2, SECOND ROW), JULY 1 (SEASON-3, THIRD ROW), AND OCTOBER 1 (SEASON-4, FOURTH ROW) OF AMES HOURLY PRECIPITATION SERIES, WITH STORMS DEFINED AS UNINTERRUPTED SEQUENCES OF RAINY HOURS.

Chapter VIII

DISCUSSIONS OF RESULTS AND CONCLUSIONS

8.1 Discussion of results. This study refers to the intermittent process of precipitation storms, with continuous precipitation intensities $\xi_t > o$ whenever it rains or snows, at a given point. Such a series, when available, gives maximum information on the precipitation process at a given precipitation gauging station. However, the precipitation time series of very small time units, say 10 minutes or less, are rarely available, and the instantaneous intensity as a function of time during storms is even more rarely available.

The original design of precipitation observation, the development of instrumentation, and the hydrometeorological services for precipitation observation have been oriented to produce discrete time series with precipitation amounts referred to calendar time units. The precipitation data is available, in general, either as data referring to the hour or multiples of an hour, to days or multiples of a day, to months or multiples of a month, or to the year. Any approximation of a continuous time series by a discrete series means a loss of information. This loss increases with an increase of the time interval over which the precipitation is integrated or averaged. Therefore, the data currently available on precipitation always has a lesser or greater loss of information when compared with the continuous intensity series of intermittent process of storms.

The stochastic process of precipitation is treated without any basic assumption about the character of this process from the probabilistic point of view. However, two phenomenological basic hypotheses, based on experience, are made. First, the process is intermittent, with continuous values $\xi_t > o$ whenever it rains or snows. Second, the process is periodic, with the year as the basic period.

The mathematical and mathematical physical description of the stochastic process of precipitation can be treated by two approaches. The first approach is when the mûltivariate distribution of the process is found and is mathematically expressed. Then it is described. This approach poses several problems. However, it is not a difficult task to accomplish, if the process is made discrete with sufficiently long time intervals of discrete values. The second approach is to select various characteristics of the process as its descriptors. These characteristics being functions of the basic process and also random variables describe the process. The problem at hand determines which characteristics should be selected for this description. Six such characteristics being the random variables of the process have been discussed in the previous text.

Distributions of selected characteristics treated as random variables can be developed mathematically under a minimum of basic phenomological assumptions. The probability distributions of six characteristics studied are functions of two basic parameters which are deterministic in character: λ_1, as the density of storms in time, and λ_2, as the yield characteristic of storms. They are constants if a process is stationary. In the process investigated in this study they are deterministic and periodic in the four examples.

It should be stressed that many other characteristics of the basic stochastic process of storms can be found to be probabilistic in nature with their distributions dependent on λ_1 and λ_2 parameters, and functions of time.

8.2 Conclusions. The following conclusions are drawn from this study:

1. The parameters of λ_1 and λ_2 are deterministic periodic functions of time, and they follow general periodic patterns of the basic parameters, such as the interval means and the interval standard deviations. The density of precipitation, defined as the mean precipitation of an interval divided by the interval length, is the ratio λ_1/λ_2.

2. The use of λ_1 and λ_2 gives a better description of the character of precipitation (with λ_1 the number of storms in a time unit, and λ_2 the inverse of the average yield per storm at a given time of the year) than the means and standard deviations of individual intervals.

3. The use of hourly and daily data for the definition of storms either over-estimates or under-estimates the number of storms per time interval. This indirectly affects the estimates of time duration of storms.

4. The number of storms in a time interval is Poisson distributed, if storms are properly defined.

5. The empirical distributions of the total precipitation for a given number of storms closely follows the theoretical distribution function derived in this paper.

6. The observed lapse time, for the given reference time of the year, of the first, second, or any other storm counted from that reference time, closely follows functions developed in this analysis.

7. The use of precipitation series in the form of hourly or daily values, or values of similar units, represents a loss of information about storms. The use of a much smaller time unit with discrete precipitation values or the use of continuous intensities during the storms may be important in the case of estimates relating to such problems as floods from small watersheds.

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APPENDICES

A - C

Proof of 2.21:

By the definition (see 2.9)

$$E_j^{t_o,t} = \{\tau_j \le t < \tau_{j+1}\}$$

It is easy to see that it can be written as follows:

$$E_j^{t_o,t} = \{\tau_j \le t\} - \{\tau_{j+1} \le t\}$$

or

$$P(E_j^{t_o,t}) = P\{\tau_j \le t\} - P\{\tau_{j+1} \le t\}$$

Taking the sum from j = 0 to j = ν - 1 of the left and right side of the last equation, we have

$$\sum_{j=o}^{\nu-1} P(E_j^{t_o,t}) = \sum_{j=o}^{\nu-1} P\{\tau_j \le t\} - \sum_{j=1}^{\nu-1} P\{\tau_{j+1} \le t\}$$

$$= P\{\tau_o \le t\} - P\{\tau_\nu \le t\}$$

Since $P\{\tau_o \le t\} = 1$ and $P\{\tau_\nu \le t\} = F_\nu(t)$, the assertion follows.

Proof of 2.22:

Suppose that the following conditions are satisfied

$$\text{(a)} \quad \lim_{\Delta t \to o} \frac{\sum_{r=2}^{\infty} P(E_r^{t,t+\Delta t})}{\Delta t} = 0 \qquad t \ge t_o$$

$$\text{(b)} \quad \lim_{\Delta t \to o} \frac{P\{E_1^{t,t+\Delta t} | E_\nu^{t_o,t}\}}{\Delta t} = \lambda_1(t,\nu) \quad t \ge t_o$$

ν = 1, 2, ... , then (2.22) follows.

Let us first consider the following relation:

$$F_\nu(t+\Delta t) = 1 - \sum_{j=o}^{\nu-1} P(E_j^{t_o,t+\Delta t}) =$$

$$1 - \sum_{j=o}^{\nu-1} \sum_{r=o}^{j} P(E_{j-r}^{t_o,t} \cap E_r^{t,t+\Delta t})$$

then on the basis of condition (a) we have

$$F_\nu(t+\Delta t) = 1 - \sum_{j=o}^{\nu-1} P(E_j^{t_o,t} \cap E_o^{t,t+\Delta t})$$

$$- \sum_{j=1}^{\nu-1} P(E_{j-1}^{t_o,t} \cap E_1^{t,t+\Delta t}) + o(\Delta t)$$

Hence

$$\frac{dF_\nu(t)}{dt}\Delta t = \sum_{j=o}^{\nu-1} P(E_j^{t_o,t} - E_o^{t,t+\Delta t} \cap E_j^{t_o,t})$$

$$- \sum_{j=o}^{\nu-2} P(E_j^{t_o,t} \cap E_1^{t,t+\Delta t}) + \sigma(\Delta t)$$

By virtue of the following relation

$$(E_o^{t,t+\ t})^c = \bigcup_{r=1}^{\infty} E_r^{t,t+\Delta t}$$

It follows

$$\frac{dF_\nu(t)}{dt}\Delta t = \sum_{j=o}^{\nu-1} P[E_j^{t_o,t} \cap (E_o^{t,t+\Delta t})^c]$$

$$- \sum_{j=o}^{\nu-2} P(E_j^{t_o,t} \cap E_1^{t,t+\Delta t}) + \sigma(\Delta t)$$

$$= P(E_{\nu-1}^{t_o,t} \cap E_1^{t,t+\Delta t}) + o(\Delta t)$$

$$= P(E_{\nu-1}^{t_o,t})\, P(E_1^{t,t+\Delta t} | E_{\nu-1}^{t_o,t}) + o(\Delta t)$$

which proves the assertion.

Proof of (2.33) and (2.34) is identical to the proof of (2.21) and (2.22), respectively.

Proof of 2.43:

Let $\eta(t_o,t)$ stand for the number of storms in (t_o,t), i.e.

$$P\{\eta(t_o,t) = \nu\} = E_\nu^{t_o,t}$$

then

$$\eta(t_o,t+\Delta t) = \eta(t_o,t) + \eta(t,t+\Delta t)$$

Taking the mathematical expectation of the left and right side of the relation, we obtain

$$E\{\eta(t_o,t+\Delta t) = E\{\eta(t_o,t)\} + E\{\eta(t,t+\Delta t)\}$$

and the assertion holds.

In a similar way one can prove (2.44).

APPENDIX B

Proof of (2.48):

By virtue of (2.32) we have

$$E(\tau_\nu) = \frac{1}{\Gamma(\nu)} \int_{t_o}^{\infty} t\, \lambda_1(s) \exp\{-\int_{t_o}^{t} \lambda_1(s)ds\}(\int_{t_o}^{t} \lambda_1(s)ds)^{\nu-1}dt$$

After partial of integration, it follows that

$$\geq \frac{1+}{\Gamma(\nu)\overline{\lambda}_1} \int_{t_o}^{\infty} \lambda_1(t) \exp\{-\int_{t_o}^{t} \lambda_1(s)ds\}(\int_{t_o}^{t} \lambda_1(s)ds)^{\nu-1}dt \geq \frac{1}{\overline{\lambda}_1}$$

$$E(\tau_\nu) = \frac{1}{\Gamma(\nu)} \int_{t_o}^{\infty} \exp\{-\int_{t_o}^{t} \lambda_1(s)ds\}(\int_{t_o}^{t} \lambda_1(s)ds)^{\nu-1}dt + E(\tau_{\nu-1})$$

Hence

$$E(\tau_\nu) - E(\tau_{\nu-1}) = \frac{1}{\Gamma(\nu)} \int_{t_o}^{\infty} \exp\{-\int \lambda_1(s)ds\}(\int_{t_o}^{t} \lambda_1(s)ds)^{\nu-1}dt$$

Similarly

$$E(\tau_\nu)-E(\tau_{\nu-1}) \leq \frac{1}{\underline{\lambda}_1\Gamma(\nu)} \int_{t_o}^{\infty} \lambda_1(t)\exp\{-\int_{t_o}^{t} \lambda_1(s)ds\} \cdot$$

$$(\int_{t_o}^{t} \lambda_1(s)ds)^{\nu-1} \leq \frac{1}{\underline{\lambda}_1}$$

Therefore

$$\frac{1}{\overline{\lambda}_1} \leq E(\tau_\nu) - E(\tau_{\nu-1}) \leq \frac{1}{\underline{\lambda}_1}$$

$$\sum_{j=1}^{\nu} \frac{1}{\overline{\lambda}_1} \leq \sum_{i=\nu}^{\nu} [E(\tau_j) - E(\tau_{j-1})] \leq \sum_{i=1}^{\nu} \frac{1}{\underline{\lambda}_1}$$

Since $E(\tau_o) = o$ we have

$$\frac{\nu}{\overline{\lambda}_1} \leq E(\tau_\nu) \leq \frac{\nu}{\underline{\lambda}_1}$$

and the assertion holds.

In the same way, one can prove (2.59).

Proof of (2.69):

$$F_t(x) = \int_\Omega EP\{x_t \leq x | \eta(t_o,t)\} dP$$

when $EP\{x_t \leq x | \eta(t_o,t)\}$ denotes the conditional probability with respect to the random variable $\eta(t_o,t)$. Since

$$X_t = \sum_{k=o}^{\eta(t_o,t)} Z_k + x_o$$

we have

$$F_t(x) = \sum_{\nu=o}^{\infty} \int_{E_\nu^{t_o,t}} P\{\sum_{k=o}^{\eta(t_o,t)} Z_k \leq x - x_o | \eta(t_o,t)\} dP$$

$$= \sum_{\nu=o}^{\infty} \int_{E_\nu^{t_o,t}} P\{\sum_{k=o}^{\nu} Z_k \leq x - x_o | \eta(t_o,t)\} dP$$

because on the set $E_\nu^{t_o,t}$ the random variable $\eta(t_o,t) = \nu$. After integration

$$F_t(x) = \sum_{\nu=o}^{\infty} P\{X_\nu \leq x | E_\nu^{t_o,t}\} P(E_\nu^{t_o,t})$$

Since

$$\{X_\nu \leq x\} = \bigcup_{j=\nu}^{\infty} G_j^{x_o,x}$$

$$F_t(x) = \sum_{\nu=o}^{\infty} P\{\bigcup_{i=\nu}^{\infty} G_j^{x_o,x} \cap E_\nu^{t_o,t}\}$$

$$= \sum_{\nu=o}^{\infty} \sum_{i=\nu}^{\infty} P(G_i^{x_o,x} \cap E_\nu^{t_o,t})$$

Proof of (2.70):

If one assumes that $G_i^{x_o,x}$ and $E_\nu^{t_o,t}$ are independent events for all $i = 0,1, \ldots$ and $\nu = o,1, \ldots$, then

$$F_t(x) = \sum_{\nu=o}^{\infty} \sum_{j=\nu}^{\infty} P(G_j^{x_o,x}) P(E_\nu^{t_o,t})$$

If in (2.32) and 2.41) we set $\lambda_1(t) = \lambda_1 =$ const. and $\lambda_2(x) = \lambda_2 =$ const. and $t_o = x_o = o$, we obtain

$$P(E_\nu^{o,t}) = e^{-\lambda_1 t} \frac{(\lambda_1 t)^\nu}{\nu!} \; ; \quad P(G_\nu^{o,x}) = e^{-\lambda_2 x} \frac{(\lambda_2 x)^\nu}{\nu!}$$

Therefore

$$F_t(x) = \sum_{\nu=o}^{\infty} \sum_{i=\nu}^{\infty} e^{-\lambda_1 t} \frac{(\lambda_1 t)^\nu}{\nu!} e^{-\lambda_2 x} \frac{(\lambda_2 x)^j}{j!}$$

$$= e^{-(\lambda_1 t + \lambda_2 x)} \sum_{\nu=o}^{\infty} \sum_{i=\nu}^{\infty} \frac{(\lambda_1 t)^\nu}{\nu!} \frac{(\lambda_2 x)^i}{i!}$$

and the assertion holds.

Key Words: Hydrology, Precipitation, Storms, Time Series, Stochastic Processes in Hydrology, Precipitation Intensities

Abstract: The continuous process of precipitation intensities, $\xi_t \geq o$, is investigated through the study of probability distributions of six descriptors: number of storms in an inverval of time, number of storms producing a given amount of precipitation, lapse time between a reference time and the end of a storm, the total precipitation of ν storms, the precipitation of ν-th storm, and the total precipitation in a time interval. The parameters λ_1, as the number of storms per time unit, and λ_2 as the inverse of the average yield per storm, are derived as periodic functions of time inside the year. The comparison of derived theoretical probability distributions, which are functions of λ_1 or λ_2, and the observed frequency distributions for the four examples used in the study, is shown to be good in the light of inevitable sampling errors.

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